8

Unruh effect

Summary Uniformly accelerated observer. The Rindler spacetime. The Rindler and Minkowski vacua. The Unruh temperature.

W. G. Unruh discovered that an observer accelerating in the Minkowski vacuum sees particles which have a thermal spectrum, with the temperature being proportional to the acceleration. This effect is called the *Unruh effect*, and in this chapter we shall derive it in a simplified case. We shall consider a massless scalar field and assume that the observer moves with a constant acceleration in a 1+1-dimensional spacetime.

8.1 Accelerated motion

The metric of the two-dimensional Minkowski spacetime is

$$ds^2 = dt^2 - dx^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta, \tag{8.1}$$

where the Greek indices α, β run over the values 0 and 1. If we use the proper time τ to parametrize the observer's trajectory $x^\alpha(\tau)$, then the 2-velocity

$$u^\alpha(\tau) \equiv \frac{dx^\alpha(\tau)}{d\tau} = \left(\dot{t}(\tau), \dot{x}(\tau)\right) \tag{8.2}$$

satisfies the normalization condition

$$\eta_{\alpha\beta} u^\alpha u^\beta = \eta_{\alpha\beta} \dot{x}^\alpha(\tau) \dot{x}^\beta(\tau) = 1, \tag{8.3}$$

where the dot denotes the derivative with respect to the proper time τ. Taking the time derivative of (8.3), we find that the 2-acceleration $a^\alpha(\tau) \equiv \dot{u}^\alpha(\tau)$ is orthogonal to the velocity,

$$\eta_{\alpha\beta} a^\alpha u^\beta = 0. \tag{8.4}$$

To understand the physical meaning of constant acceleration, one can imagine a spaceship with a propulsion engine that exerts a constant force. In this case, we intuitively expect that the acceleration of the spaceship is constant. In an instantly comoving inertial frame the observer is at rest and $\dot{x}(\tau) = 0$; hence $u^\alpha(\tau) = (1, 0)$. It then follows from (8.4) that $a^\alpha(\tau) = (0, a)$, where $a = \text{const}$. Since this is valid at any moment of time τ, the condition of constant acceleration can be formulated in the following completely covariant form, applicable in *any* (not necessarily comoving) inertial frame:

$$\eta_{\alpha\beta} a^\alpha(\tau) a^\beta(\tau) = \eta_{\alpha\beta} \ddot{x}^\alpha(\tau) \ddot{x}^\beta(\tau) = -a^2. \tag{8.5}$$

We have seen in the previous chapters that the notion of a particle crucially depends on the definition of the positive-frequency modes. For an inertial observer, these modes should be defined with respect to the time t of some inertial frame (the notion of a particle is Lorentz-invariant). However, when we consider an accelerated observer it is natural to expect that the positive-frequency modes must be defined with respect to the proper time of this observer. Then a "particle" registered by an accelerated detector can be very different from a "particle" of an inertial observer. As a result an accelerated observer views the Minkowski vacuum as a state containing particles. To verify this statement we need to (a) determine the trajectory of the accelerated observer in an inertial frame, (b) construct an accelerated comoving coordinate frame, and finally (c) solve the wave equation and compare the definition of particles in both coordinate frames. All these steps are significantly simplified if we use the lightcone coordinates instead of t and x.

Lightcone coordinates The inertial lightcone coordinates are defined as

$$u \equiv t - x, \qquad v \equiv t + x \tag{8.6}$$

and metric (8.1) then becomes

$$ds^2 = du\, dv = g^{(M)}_{\alpha\beta} dx^\alpha dx^\beta, \tag{8.7}$$

where $x^0 \equiv u$, $x^1 \equiv v$ and

$$g^{(M)}_{\alpha\beta} = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix} \tag{8.8}$$

is the Minkowski metric in the lightcone coordinates. One can easily see that the coordinate transformation

$$u \to \tilde{u} = \alpha u, \; v \to \tilde{v} = \frac{1}{\alpha} v, \tag{8.9}$$

with $\alpha = \text{const}$, preserves metric (8.7) and therefore corresponds to a Lorentz transformation.

Exercise 8.1
Find an expression for α in terms of the relative velocity of the two inertial frames.

The trajectory of an accelerated observer Let us now determine the trajectory of a uniformly accelerated observer in the inertial frame. In the lightcone coordinates this trajectory is described by

$$x^{\alpha}(\tau) = (u(\tau), v(\tau)). \tag{8.10}$$

Replacing $\eta_{\alpha\beta}$ by $g^{(M)}_{\alpha\beta}$ in (8.3), (8.5) and substituting (8.10) for $x^{\alpha}(\tau)$ we obtain

$$\dot{u}(\tau)\dot{v}(\tau) = 1, \tag{8.11}$$

$$\ddot{u}(\tau)\ddot{v}(\tau) = -a^2. \tag{8.12}$$

Since $\dot{u}(\tau) = 1/\dot{v}(\tau)$, we have

$$\ddot{u} = -\frac{\ddot{v}}{\dot{v}^2},$$

and (8.12) reduces to

$$\left(\frac{\ddot{v}}{\dot{v}}\right)^2 = a^2.$$

This equation can be easily integrated, yielding the result

$$v(\tau) = \frac{A}{a}e^{a\tau} + B,$$

where A and B are integration constants. It then follows from $\dot{u}(\tau) = 1/\dot{v}(\tau)$ that

$$u(\tau) = -\frac{1}{Aa}e^{-a\tau} + C,$$

where C is a further integration constant. Performing a Lorentz transformation (8.9), one can set $A = 1$. Furthermore, shifting the origin of the corresponding inertial frame we can make both the integration constants B and C vanish, and then the trajectory of the accelerated observer is described by

$$u(\tau) = -\frac{1}{a}e^{-a\tau}, \qquad v(\tau) = \frac{1}{a}e^{a\tau}. \tag{8.13}$$

Using definitions (8.6) and going back to the original Minkowski coordinates t and x we obtain

$$t(\tau) = \frac{v+u}{2} = \frac{1}{a}\sinh a\tau, \qquad x(\tau) = \frac{v-u}{2} = \frac{1}{a}\cosh a\tau. \tag{8.14}$$

Thus, the worldline of the accelerated observer is a branch of the hyperbola $x^2 - t^2 = a^{-2}$ in the (t, x) plane (see Fig. 8.1). For large $|t|$, the worldline approaches the lightcone. The observer arrives from $x = +\infty$, decelerates and stops at $x = a^{-1}$, then accelerates back towards infinity.

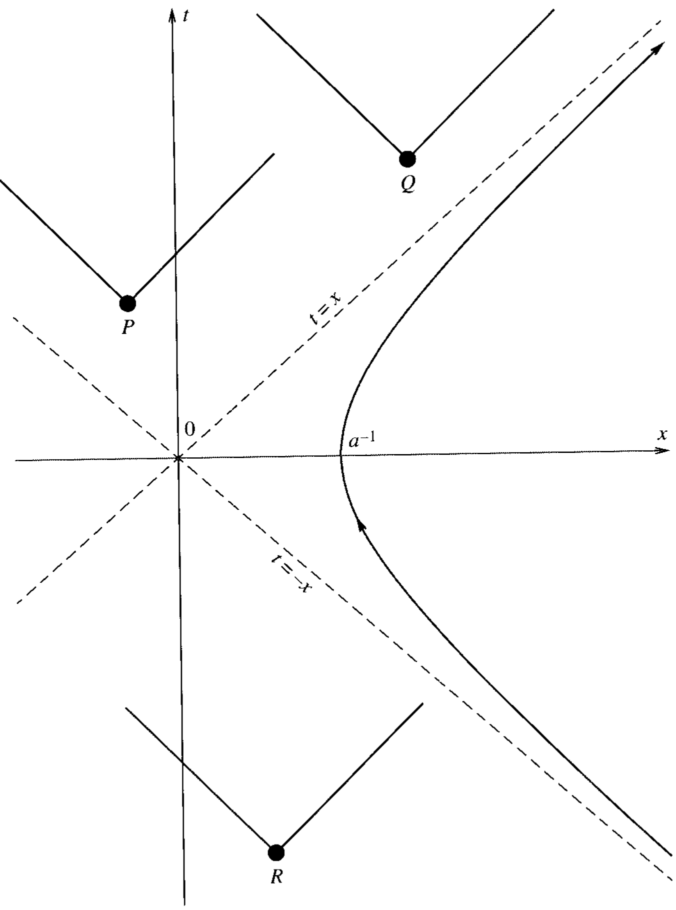

Fig. 8.1 The worldline of a uniformly accelerated observer (proper acceleration $a \equiv |\mathbf{a}|$) in Minkowski spacetime. The dashed lines show the lightcone. The observer cannot receive any signals from the events P, Q and cannot send signals to R.

8.2 Comoving frame of accelerated observer

As the next step, we will find an appropriate comoving frame (ξ^0, ξ^1) for an accelerated observer. We are looking for a coordinate system in which the observer is at rest at $\xi^1 = 0$. In addition, the time coordinate ξ^0 must coincide with the proper time τ along the observer's worldline. Finally, we would like the metric in the comoving frame to be *conformally flat*,

$$ds^2 = \Omega^2(\xi^0, \xi^1)\left[(d\xi^0)^2 - (d\xi^1)^2\right], \tag{8.15}$$

where $\Omega(\xi^0, \xi^1)$ is a function to be determined. The conformally flat form of the metric greatly simplifies quantization of fields. Our present task is to show that such a coordinate system can be found and to establish the relation between ξ^0, ξ^1 and the Minkowski coordinates t, x.

It is convenient to use the lightcone coordinates of the comoving frame,

$$\tilde{u} \equiv \xi^0 - \xi^1, \tilde{v} \equiv \xi^0 + \xi^1, \tag{8.16}$$

where metric (8.15) takes the form

$$ds^2 = \Omega^2(\tilde{u}, \tilde{v})\, d\tilde{u}\, d\tilde{v}. \tag{8.17}$$

In terms of the lightcone coordinates, the observer's worldline,

$$\xi^0(\tau) = \tau, \quad \xi^1(\tau) = 0, \tag{8.18}$$

takes the form

$$\tilde{v}(\tau) = \tilde{u}(\tau) = \tau. \tag{8.19}$$

Since ξ^0 is the proper time at the observer's location, the conformal factor $\Omega^2(\tilde{u}, \tilde{v})$ must satisfy

$$\Omega^2(\tilde{u} = \tau, \tilde{v} = \tau) = 1. \tag{8.20}$$

Metrics (8.7) and (8.17) describe the same Minkowski spacetime in different coordinate systems and therefore

$$ds^2 = du\, dv = \Omega^2(\tilde{u}, \tilde{v})\, d\tilde{u}\, d\tilde{v}. \tag{8.21}$$

The functions $u(\tilde{u}, \tilde{v})$ and $v(\tilde{u}, \tilde{v})$ can depend only on one of two arguments, either $\tilde{u}$ or $\tilde{v}$; otherwise there will arise the terms $d\tilde{u}^2$ and $d\tilde{v}^2$ in the latter equality in (8.21). To be definite, let us choose

$$u = u(\tilde{u}), \qquad v = v(\tilde{v}). \tag{8.22}$$

We shall now determine the required functions $u(\tilde{u})$ and $v(\tilde{v})$.

Considering the observer's trajectory in two coordinate systems, we have

$$\frac{du(\tau)}{d\tau} = \frac{du(\tilde{u})}{d\tilde{u}} \frac{d\tilde{u}(\tau)}{d\tau}. \tag{8.23}$$

It follows from (8.13) and (8.19) that

$$\frac{du(\tau)}{d\tau} = e^{-a\tau} = -au(\tau), \quad \frac{d\tilde{u}(\tau)}{d\tau} = 1,$$

and (8.23) reduces to the following equation for $u(\tilde{u})$:

$$\frac{du}{d\tilde{u}} = -au; \tag{8.24}$$

hence

$$u = C_1 e^{-a\tilde{u}},$$

where C_1 is an integration constant. Similarly we find

$$v = C_2 e^{a\tilde{v}}.$$

Condition (8.20) restricts integration constants to satisfy $a^2 C_1 C_2 = -1$. Taking $C_2 = -C_1$ we obtain

$$u = -\frac{1}{a} e^{-a\tilde{u}}, \qquad v = \frac{1}{a} e^{a\tilde{v}}, \tag{8.25}$$

and

$$ds^2 = du\, dv = e^{a(\tilde{v}-\tilde{u})} d\tilde{u}\, d\tilde{v}. \tag{8.26}$$

With the help of the definitions in (8.6) and (8.16), one can rewrite relations (8.25) in terms of t, x and ξ^0, ξ^1:

$$t\left(\xi^0, \xi^1\right) = a^{-1} e^{a\xi^1} \sinh a\xi^0, \qquad x\left(\xi^0, \xi^1\right) = a^{-1} e^{a\xi^1} \cosh a\xi^0. \tag{8.27}$$

The metric in accelerated frame,

$$ds^2 = e^{2a\xi^1} \left[\left(d\xi^0\right)^2 - \left(d\xi^1\right)^2 \right], \tag{8.28}$$

describes the *Rindler spacetime*, which is locally equivalent to Minkowski spacetime and therefore has zero curvature. In Fig. 8.2 we show the hypersurfaces $\xi^0 = \text{const}$ and $\xi^1 = \text{const}$ in the (t, x) plane. The coordinates ξ^0, ξ^1 span the ranges

$$-\infty < \xi^0 < +\infty, \qquad -\infty < \xi^1 < +\infty,$$

covering only one quarter of the Minkowski spacetime (the domain $x > |t|$). Hence, the coordinate system (ξ^0, ξ^1) is *incomplete*. The accelerated observer cannot measure distances larger than a^{-1} in the direction opposite to the acceleration. To see this, let us consider a hypersurface of constant time $\xi^0 = \text{const}$. An infinite range of the spacelike coordinate, $-\infty < \xi^1 < 0$, where $\xi^1 = 0$ is the observer's location, spans a *finite* physical distance,

$$d = \int_{-\infty}^{0} e^{a\xi^1} d\xi^1 = \frac{1}{a}.$$

Therefore, there is no comoving accelerated frame which could cover the entire Minkowski spacetime. The lightcone $t = x$ plays the role of the event horizon; for example, the events P and Q will never become visible to the accelerated observer.

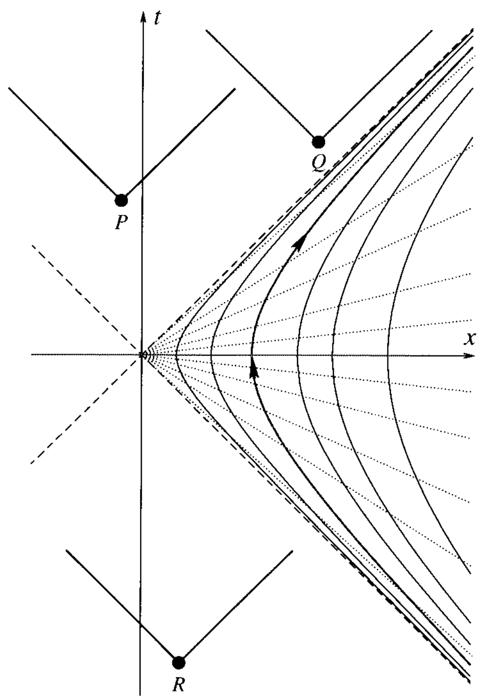

Fig. 8.2 The proper coordinate system of a uniformly accelerated observer in Minkowski spacetime. The solid hyperbolae are the lines of constant proper distance ξ^1; the hyperbola with arrows is $\xi^1 = 0$, or $x^2 - t^2 = a^{-2}$. The lines of constant ξ^0 are dotted. The dashed lines show the lightcone which corresponds to $\xi^1 = -a^{-1}$. The events P, Q, R are not covered by the proper coordinate system.

8.3 Quantum fields in inertial and accelerated frames

What a particular detector registers as a particle depends on the clocks used: Particles are determined by the positive frequency modes with respect to the proper time of the observer. An inertial observer defines these modes using the Minkowski time t, while the accelerated observer must use the proper time $\tau = \xi^0$. Since t and ξ^0 are related in a nontrivial way, one expects that the positive frequency mode with respect to t is a superposition of the positive and negative frequency modes with respect to ξ^0. As a result the Minkowski vacuum can appear as a state containing particles from the point of view of the accelerated observer.

The goal of this section is to show that this really takes place. The problem is greatly simplified for a massless scalar field in 1+1-dimensional spacetime because the action

$$S[\phi] = \frac{1}{2} \int g^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} \sqrt{-g} d^2 x \tag{8.29}$$

is conformally invariant. Under a conformal transformation

$$g_{\alpha\beta} \to \tilde{g}_{\alpha\beta} = \Omega^2(x^\gamma) g_{\alpha\beta},$$

the determinant $\sqrt{-g}$ and the contravariant metric change as

$$\sqrt{-g} \to \sqrt{-\tilde{g}} = \Omega^2 \sqrt{-g}, \qquad g^{\alpha\beta} \to \tilde{g}^{\alpha\beta} = \Omega^{-2} g^{\alpha\beta}, \tag{8.30}$$

and the factors Ω^2 cancel in the action. (Note that a minimally coupled massless scalar field in 3+1 dimensions is *not* conformally invariant.) This explains why the action looks similar in both the inertial and the accelerated frames:

$$\begin{aligned} S &= \frac{1}{2} \int \left[(\partial_t \phi)^2 - (\partial_x \phi)^2 \right] dt dx \\ &= \frac{1}{2} \int \left[(\partial_{\xi^0} \phi)^2 - (\partial_{\xi^1} \phi)^2 \right] d\xi^0 d\xi^1. \end{aligned} \tag{8.31}$$

Rewritten in terms of the lightcone coordinates, this action takes the form

$$S = 2 \int \partial_u \phi \partial_v \phi \, du dv = 2 \int \partial_{\tilde{u}} \phi \partial_{\tilde{v}} \phi \, d\tilde{u} d\tilde{v}.$$

The field equations

$$\partial_u \partial_v \phi = 0, \qquad \partial_{\tilde{u}} \partial_{\tilde{v}} \phi = 0,$$

have simple solutions,

$$\phi(u, v) = A(u) + B(v), \quad \phi(\tilde{u}, \tilde{v}) = \tilde{A}(\tilde{u}) + \tilde{B}(\tilde{v}), \tag{8.32}$$

where A, $\tilde{A}$ etc. are arbitrary smooth functions. In particular,

$$\phi \propto e^{-i\omega u} = e^{-i\omega(t-x)}$$

describes a right-moving, positive-frequency mode with respect to the Minkowski time t, while

$$\phi \propto e^{-i\Omega \tilde{u}} = e^{-i\Omega(\xi^0 - \xi^1)}$$

corresponds to a right-moving positive-frequency mode with respect to $\tau = \xi^0$. The solutions $\phi \propto e^{-i\omega v}$ and $\phi \propto e^{-i\Omega \tilde{v}}$ describe left-moving modes. Since $u = u(\tilde{u})$ and $v = v(\tilde{v})$, and the solutions of the wave equation have the form (8.32), the left- and right-moving modes do not affect each other and can be considered separately. In all formulae below we write explicitly only the right-moving modes. The reader can easily recover the contributions of the left-moving modes.

The actions in (8.31) have a canonical form. Therefore, in the domain $x > |t|$ of the spacetime where both coordinate frames overlap, we can immediately write the standard mode expansions for the field operator $\hat{\phi}$ as

$$\hat{\phi} = \int_0^\infty \frac{d\omega}{(2\pi)^{1/2}} \frac{1}{\sqrt{2\omega}} \left[e^{-i\omega u} \hat{a}_\omega^- + e^{i\omega u} \hat{a}_\omega^+ \right] + \text{(left-moving)}$$

$$= \int_0^\infty \frac{d\Omega}{(2\pi)^{1/2}} \frac{1}{\sqrt{2\Omega}} \left[e^{-i\Omega \tilde{u}} \hat{b}_\Omega^- + e^{i\Omega \tilde{u}} \hat{b}_\Omega^+ \right] + \text{(left-moving)}, \tag{8.33}$$

where both sets of operators $\hat{a}_\omega^\pm$ and $\hat{b}_\Omega^\pm$ satisfy the standard commutation relations:

$$\left[\hat{a}_\omega^-, \hat{a}_{\omega'}^+ \right] = \delta(\omega - \omega'), \quad \left[\hat{b}_\Omega^-, \hat{b}_{\Omega'}^+ \right] = \delta(\Omega - \Omega'), \text{ etc.} \tag{8.34}$$

Because we consider one spatial dimension, the normalization factor $(2\pi)^{1/2}$ in these formulae replaces the factor $(2\pi)^{3/2}$ used in the three-dimensional case.

The *Minkowski vacuum* $|0_M\rangle$ is the zero eigenvector of all annihilation operators $\hat{a}_\omega^-$, that is,

$$\hat{a}_\omega^- |0_M\rangle = 0.$$

In turn, the operators $\hat{b}_\Omega^-$ define the *Rindler vacuum* state $|0_R\rangle$,

$$\hat{b}_\Omega^- |0_R\rangle = 0.$$

The corresponding particle states are then built with the help of the creation operators in the standard way.

The states $|0_M\rangle$ and $|0_R\rangle$ are different and a natural question to ask is which of them is the "correct" vacuum. The answer to this question depends on the particular physical experiment considered. For example, for normalization of energy which contributes to the gravitational field we have to use the Minkowski vacuum $|0_M\rangle$. On the other hand, the detector of the accelerated observer reacts to particles associated with the Rindler vacuum $|0_R\rangle$. The detector remains unexcited only if the quantum field is in the state $|0_R\rangle$, while the Minkowski vacuum $|0_M\rangle$ is, from the point of view of the accelerated observer, a state containing particles. This is a manifestation of the Unruh effect.

In the rest of this chapter we find the relation between operators $\hat{a}_k^\pm$ and $\hat{b}_k^\pm$ and calculate the occupation numbers of the Rindler particles in the Minkowski vacuum state.

Remark: Rindler vacuum In contrast to the Minkowski vacuum, the Rindler vacuum is an unphysical state which is singular on the horizons $u = 0$ and $v = 0$. To get a rough idea why this is so, let us consider the appropriately regularized expectation values of the operators $(\partial_u\hat{\phi})^2$ and $(\partial_{\tilde{u}}\hat{\phi})^2$ for the Minkowski and Rindler vacuum states respectively. It follows from the expansions in (8.33) that

$$\langle 0_M| \left(\partial_u\hat{\phi}\right)^2 |0_M\rangle = \langle 0_R| \left(\partial_{\tilde{u}}\hat{\phi}\right)^2 |0_R\rangle, \tag{8.35}$$

and as a result we have

$$\langle 0_R| \left(\partial_u\hat{\phi}\right)^2 |0_R\rangle = \left(\frac{\partial\tilde{u}}{\partial u}\right)^2 \langle 0_R| \left(\partial_{\tilde{u}}\hat{\phi}\right)^2 |0_R\rangle = \frac{1}{a^2u^2} \langle 0_M| \left(\partial_u\hat{\phi}\right)^2 |0_M\rangle, \tag{8.36}$$

where (8.25) has been used. Thus, the expectation values of $(\partial_u\hat{\phi})^2$ taken for the Rindler and Minkowski vacua are related by a coordinate-dependent factor which becomes infinite on the future horizon $u = 0$. Because the Minkowski vacuum is a physically well-defined state, the Rindler vacuum is a singular state which requires an infinite energy to be prepared. Note that only the right-moving modes contribute to $(\partial_u\hat{\phi})^2$ and they are responsible for the singularity of the Rindler vacuum on the future horizon. Similarly, by considering $(\partial_v\hat{\phi})^2$ and $(\partial_{\tilde{v}}\hat{\phi})^2$, we can find that the left-moving modes lead to a singularity of the Rindler vacuum on the past horizon at $v = 0$.

8.4 Bogolyubov transformations

The operators $\hat{a}^{\pm}$ and $\hat{b}^{\pm}$ are related by the Bogolyubov transformations

$$\hat{b}_{\Omega}^{-} = \int_0^{\infty} d\omega \left[\alpha_{\Omega\omega}\hat{a}_{\omega}^{-} - \beta_{\Omega\omega}\hat{a}_{\omega}^{+}\right]. \tag{8.37}$$

Because the Rindler coordinates cover only a quarter of Minkowski spacetime, the inverse Bogolyubov transformation is not defined. The transformations (8.37) have a more general form than in (6.28) because *all* positive and negative frequency modes with respect to t contribute to the positive frequency mode with respect to τ, whereas the Bogolyubov transformations in (6.28) are "diagonal," with $\alpha_{\omega\Omega}$ and $\beta_{\omega\Omega}$ proportional to $\delta(\omega - \Omega)$. The normalization condition for the Bogolyubov coefficients,

$$\int_0^{\infty} d\omega \left(\alpha_{\Omega\omega}\alpha^*_{\Omega'\omega} - \beta_{\Omega\omega}\beta^*_{\Omega'\omega}\right) = \delta(\Omega - \Omega'), \tag{8.38}$$

follows from the compatibility of the commutation relations in (8.34). (This is the generalization of the condition $|\alpha_{\mathbf{k}}|^2 - |\beta_{\mathbf{k}}|^2 = 1$ we encountered before.)

Exercise 8.2
Derive (8.38).

Substituting (8.37) into (8.33) we infer that

$$\frac{1}{\sqrt{\omega}}e^{-i\omega u} = \int_0^\infty \frac{d\Omega'}{\sqrt{\Omega'}}\left(\alpha_{\Omega'\omega}e^{-i\Omega'\tilde{u}} - \beta^*_{\Omega'\omega}e^{i\Omega'\tilde{u}}\right). \tag{8.39}$$

Multiplying both sides of this relation by $\exp(\pm i\Omega\tilde{u})$, taking into account that

$$\int_{-\infty}^{+\infty} e^{i(\Omega-\Omega')\tilde{u}}d\tilde{u} = 2\pi\delta(\Omega-\Omega'),$$

and integrating over $\tilde{u}$ we obtain the result in terms of the Γ-function:

$$\left\{\begin{matrix}\alpha_{\Omega\omega} \\ \beta_{\Omega\omega}\end{matrix}\right. = \int_{-\infty}^{+\infty} e^{\mp i\omega u + i\Omega\tilde{u}}d\tilde{u} = \pm\frac{1}{2\pi}\sqrt{\frac{\Omega}{\omega}}\int_{-\infty}^{0}(-au)^{-\frac{i\Omega}{a}-1}e^{\mp i\omega u}du \tag{8.40}$$

$$= \pm\frac{1}{2\pi a}\sqrt{\frac{\Omega}{\omega}}e^{\pm\frac{\pi\Omega}{2a}}\exp\left(\frac{i\Omega}{a}\ln\frac{\omega}{a}\right)\Gamma\left(-\frac{i\Omega}{a}\right). \tag{8.41}$$

It follows that α and β obey the useful relation,

$$|\alpha_{\Omega\omega}|^2 = e^{\frac{2\pi\Omega}{a}}|\beta_{\Omega\omega}|^2. \tag{8.42}$$

Exercise 8.3
Derive (8.41).

Exercise 8.4
Derive (8.42) directly from (8.40) without using the Γ-function.

8.5 Occupation numbers and Unruh temperature

The vacua $|0_M\rangle$ and $|0_R\rangle$, annihilated by the operators $\hat{a}^-_\omega$ and $\hat{b}^-_\Omega$, are different. The Minkowski a-vacuum is a state with Rindler b-particles and vice versa. We now compute the number of b-particles in the a-vacuum state. The expectation value of the b-particle number operator $\hat{N}_\Omega \equiv \hat{b}^+_\Omega\hat{b}^-_\Omega$ in the Minkowski vacuum $|0_M\rangle$ is

$$\langle\hat{N}_\Omega\rangle \equiv \langle 0_M|\hat{b}^+_\Omega\hat{b}^-_\Omega|0_M\rangle = \langle 0_M|\left(\int d\omega\left[\alpha^*_{\omega\Omega}\hat{a}^+_\omega - \beta^*_{\omega\Omega}\hat{a}^-_\omega\right]\right)$$
$$\times\left(\int d\omega'\left[\alpha_{\omega'\Omega}\hat{a}^-_{\omega'} - \beta_{\omega'\Omega}\hat{a}^+_{\omega'}\right]\right)|0_M\rangle = \int d\omega\,|\beta_{\omega\Omega}|^2, \tag{8.43}$$

and this is interpreted as the mean number of particles with frequency Ω found by the accelerated observer.

For $\Omega' = \Omega$ the normalization condition (8.38) becomes

$$\int_0^\infty d\omega \left(|\alpha_{\Omega\omega}|^2 - |\beta_{\Omega\omega}|^2 \right) = \delta(0), \tag{8.44}$$

and taking into account (8.42) we find

$$\langle \hat{N}_\Omega \rangle = \int_0^{+\infty} d\omega \, |\beta_{\omega\Omega}|^2 = \left[\exp\left(\frac{2\pi\Omega}{a}\right) - 1 \right]^{-1} \delta(0). \tag{8.45}$$

The divergent factor $\delta(0)$ is due to the infinite volume of the entire space. If the field were quantized in a finite box of volume V, the momenta ω and Ω would be discrete and $\delta(0)$ would be replaced by the volume V, that is, $\delta(0) = V$. Thus the mean density of the particles with frequency Ω is

$$n_\Omega = \frac{\langle \hat{N}_\Omega \rangle}{V} = \left[\exp\left(\frac{2\pi\Omega}{a}\right) - 1 \right]^{-1}. \tag{8.46}$$

This is the main result of this chapter.

We have computed n_Ω only for right-moving modes (with positive momenta). The result for left-moving modes is obtained similarly. Massless particles detected by the accelerated detector in the Minkowski vacuum obey the Bose–Einstein distribution (8.46) with the *Unruh temperature*

$$T \equiv \frac{a}{2\pi}. \tag{8.47}$$

Thus an accelerated observer will see a thermal bath of particles. A physical interpretation of the Unruh effect is the following. The accelerated detector is coupled to the quantum vacuum fluctuations and these fluctuations act on the detector and excite it as if the detector were in a thermal bath with the temperature proportional to the acceleration. However, vacuum fluctuations in Minkowski spacetime cannot supply their own energy to excite the detector and they serve only as a "mediator" borrowing the energy from the agent responsible for acceleration. The acceleration required to produce a measurable temperature is enormous and therefore it is unlikely that the Unruh effect can be verified in the near future (see Exercise 1.6 on p. 12 for a numerical example). The energy spent by the accelerating agent is exponentially large compared with the energy in detected particles.

Finally, we note that the consideration above can be straightforwardly generalized to the case of four-dimensional spacetime, as well as to spinor- and vector-valued quantum fields.