7

Quantum fields in the de Sitter universe

Summary Field quantization in de Sitter spacetime. Bunch–Davies vacuum. Time evolution of quantum fluctuations.

7.1 De Sitter universe

We now apply the formalism developed in the previous chapter to study the behavior of quantum fluctuations in the *de Sitter universe*. The de Sitter universe is a particular case of a homogeneous and isotropic universe with a positive cosmological constant Λ. Formally this cosmological constant can be thought of as an "ideal hydrodynamical fluid" with the equation of state

$$p_\Lambda = -\varepsilon_\Lambda. \tag{7.1}$$

In this case, the energy-momentum tensor of the perfect fluid becomes

$$T^\mu_\nu = (\varepsilon + p)u^\mu u_\nu - p\delta^\mu_\nu = \varepsilon_\Lambda \delta^\mu_\nu,$$

and it follows from the conservation law $T^\alpha_{\beta;\alpha} = 0$ that $\varepsilon_\Lambda = \text{const}$. This is the energy-momentum tensor corresponding to a cosmological constant. For a flat isotropic universe, the 0-0 component of the Einstein equations (called the Friedmann equation) reduces to

$$H \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\varepsilon_\Lambda, \tag{7.2}$$

where $a(t)$ is the scale factor and the dot denotes the derivative with respect to the physical time t. This equation has the obvious solution

$$a(t) = a_0 e^{H_\Lambda t}, \tag{7.3}$$

which describes a flat de Sitter universe with the time-independent Hubble parameter

$$H_\Lambda = \sqrt{\frac{8\pi G}{3}\varepsilon_\Lambda}.$$

In this case it is easy to verify that all curvature invariants are constant and therefore the metric

$$ds^2 = dt^2 - H_\Lambda^{-2}\exp(2H_\Lambda t)\,\delta_{ik}dx^i dx^k \tag{7.4}$$

(where for convenience we set $a_0 = H_\Lambda^{-1}$) describes a static maximally symmetric spacetime in expanding coordinates. There exist no static coordinates which can cover the de Sitter spacetime on scales larger that the curvature scale H_Λ^{-1}, and even the expanding coordinates in (7.4) are incomplete. To verify this, let us first rewrite metric (7.4) in terms of the conformal time

$$\eta = -\int_t^\infty \frac{dt}{a(t)} = -\exp(-H_\Lambda t),$$

instead of the physical time t and the spherical coordinates instead of x^i. The result is

$$ds^2 = \frac{1}{H_\Lambda^2\eta^2}\left[d\eta^2 - dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right], \tag{7.5}$$

where $-\infty < \eta < 0$ and $0 \le r < \infty$. Next we change from η, r to the new coordinates $\tilde{\eta}, \chi$ which are related to the old ones (in the region where both overlap) via

$$\eta = \frac{\sin\tilde{\eta}}{\cos\tilde{\eta} + \cos\chi}, \qquad r = \frac{\sin\chi}{\cos\tilde{\eta} + \cos\chi}. \tag{7.6}$$

Metric (7.5) then takes the form

$$ds^2 = \frac{1}{H_\Lambda^2 \sin^2\tilde{\eta}}\left[d\tilde{\eta}^2 - d\chi^2 - \sin^2\chi\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right], \tag{7.7}$$

and describes a closed de Sitter universe which first contracts for $-\pi < \tilde{\eta} < -\pi/2$, reaches the minimal radius at $\tilde{\eta} = -\pi/2$ and then expands so that $a \to \infty$ as $\tilde{\eta} \to -0$. It is obvious, however, that (7.7) simply corresponds to another coordinate choice for the same de Sitter spacetime. In this sense the de Sitter universe is a very special case of the Friedmann universe. Generally, the energy density in the Friedmann universe is time-dependent and the geometry of hypersurfaces of constant energy density is unambiguously determined; hence the closed and flat universes are physically distinguishable. In the de Sitter universe, however, the energy density is time-independent and therefore any hypersurface is a hypersurface of constant energy. As a consequence, the flat, closed, and open de Sitter universes describe the same spacetime in different coordinate systems. The coordinates in (7.7) span the ranges

$$-\pi < \tilde{\eta} < 0, \quad 0 \leq \chi \leq \pi,$$

covering the entire de Sitter spacetime. The nontrivial time-radial part of this spacetime can be graphically represented by a square in Fig. 7.1, called a *conformal diagram*. Note that each point of the diagram corresponds to a two-dimensional sphere and the radial null geodesics, determined by equation $ds^2 = 0$, are straight lines at $\pm 45°$ angles.

Using relations (7.6) to draw the hypersurfaces $\eta = \text{const}$ and $r = \text{const}$ in the $\tilde{\eta} - \chi$ plane, we find that the coordinates in (7.5) cover only a half of the entire de Sitter spacetime (see Fig. 7.1). Therefore, these coordinates are incomplete. In cosmological applications, however, only a relatively small region of the de Sitter spacetime (shaded in Fig. 7.1) is used to approximate the inflationary epoch in the history of the universe. Within this region the closed and flat coordinates are similar and hence the incompleteness of the flat coordinates is not a problem. On the other hand, the analysis of the behavior of quantum fields is significantly simplified in these coordinates. Therefore, we shall use the flat coordinates and ignore their inability to cover events in the distant past which are irrelevant for physical applications.

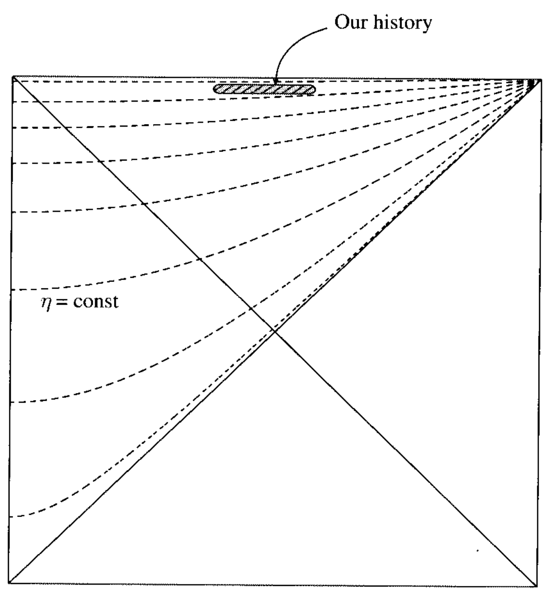

Fig. 7.1 A conformal diagram of de Sitter spacetime. The flat coordinate system covers only the left upper half of the diagram. Dashed lines are surfaces of constant η.

Remark: horizons An interesting feature of de Sitter spacetime is the presence of both particle and event horizons. Given a particular moment of cosmic time, the *particle horizon* is the boundary of the spatial region which consists of the points causally connected to an observer. Let us consider the closed de Sitter universe. Taking into account that the radial light geodesics are described by

$$\chi(\tilde{\eta}) = \pm\tilde{\eta} + \text{const},$$

we find that at time $\tilde{\eta}$ only those points which have comoving coordinates

$$\chi < \chi_p(\tilde{\eta}) = \tilde{\eta} + \pi \tag{7.8}$$

are causally connected to an observer at $\chi = 0$. The physical size of the particle horizon,

$$d_p \equiv a(\tilde{\eta})\,\chi_p(\tilde{\eta}) = -\frac{\tilde{\eta}+\pi}{H_\Lambda \sin\tilde{\eta}}, \tag{7.9}$$

grows as $d_p \propto -1/\tilde{\eta} \propto \exp(H_\Lambda t)$ when $\tilde{\eta} \to -0$.

Any event happening at time $\tilde{\eta}$ at distances

$$\chi > \chi_e(\tilde{\eta}) = -\tilde{\eta} \tag{7.10}$$

will never be seen by an observer at $\chi = 0$. The sphere with comoving radius $\chi_e(\tilde{\eta})$ is called the *event horizon*. Its physical size,

$$d_e \equiv a(\tilde{\eta})\,\chi_e(\tilde{\eta}) = \frac{\tilde{\eta}}{H_\Lambda \sin\tilde{\eta}}, \tag{7.11}$$

approaches the curvature scale H_Λ^{-1} as $\tilde{\eta} \to -0$. This limit corresponds to the exponentially expanding universe: $a \propto \exp(H_\Lambda t)$. The origin of the event horizon in the de Sitter universe is rather curious. An observer never catches lightrays emitted at a distance $d > H_\Lambda^{-1}$ because the intervening space expands too quickly.

7.2 Quantization

Now we quantize a massive scalar field $\phi(\mathbf{x}, \eta)$ with the potential $V(\phi) = \frac{1}{2}m^2\phi^2$ in the de Sitter background. Because the flat de Sitter universe is a particular case of the flat Friedmann universe, we can use the formulae from the previous chapter without any alterations by simply substituting for the scale factor

$$a(\eta) = -\frac{1}{H_\Lambda \eta}, \qquad -\infty < \eta < 0.$$

Introducing the auxiliary field $\chi \equiv a\phi$ and noting that

$$\omega_k^2(\eta) = k^2 + m^2 a^2 - \frac{a''}{a} = k^2 + \left(\frac{m^2}{H_\Lambda^2} - 2\right)\frac{1}{\eta^2}, \tag{7.12}$$

we find that the mode function satisfies (see (6.21)):

$$v_k'' + \left[k^2 - \left(2 - \frac{m^2}{H_\Lambda^2}\right)\frac{1}{\eta^2}\right]v_k = 0. \tag{7.13}$$

The general solution of this equation is given in terms of the Bessel functions $J_n(x)$ and $Y_n(x)$:

$$v_k(\eta) = \sqrt{k|\eta|}\left[A_k J_n(k|\eta|) + B_k Y_n(k|\eta|)\right], \quad n \equiv \sqrt{\frac{9}{4} - \frac{m^2}{H_\Lambda^2}} \tag{7.14}$$

(see Exercise 7.1). The normalization condition $\mathrm{Im}\left(v_k^* v_k'\right) = 1$ (see (6.22)) constrains the integration constants A_k and B_k by

$$A_k B_k^* - A_k^* B_k = \frac{i\pi}{k}.$$

Exercise 7.1

Assuming that $m/H_\Lambda < 3/2$, find a change of variables which reduces (7.13) to the Bessel equation

$$s^2\frac{d^2 f}{ds^2} + s\frac{df}{ds} + \left(s^2 - n^2\right)f = 0$$

with the general solution

$$f(s) = AJ_n(s) + BY_n(s).$$

Using the asymptotics of the Bessel functions, determine the behavior of $v_k(\eta)$ for $k|\eta| \gg 1$ and $k|\eta| \ll 1$.

The asymptotic behavior of the solutions can be found directly from equation (7.13). Given a wavenumber k, let us consider the early time asymptotic $k|\eta| \gg 1$ (which corresponds to large negative η). In this case the physical wavelength,

$$L_{\mathrm{p}} \sim a(\eta)k^{-1} \simeq \frac{H_\Lambda^{-1}}{k|\eta|}, \tag{7.15}$$

is much smaller than the curvature scale H_Λ^{-1}. Thus we expect that the corresponding mode is not affected by gravity and behaves as in Minkowski space. For $k|\eta| \gg 1$ one can neglect the η^{-2} term compared with k^2 in (7.12) and hence $\omega_k \approx k$. The two independent solutions of (7.13) are then $\propto \exp(\pm ik\eta)$ and we can define the minimal excitation ("vacuum") state for the corresponding modes by choosing the negative-frequency mode as

$$v_k(\eta) \approx \frac{1}{\sqrt{k}}e^{ik\eta}. \tag{7.16}$$

This determines the vacuum state only to the leading order in $|k\eta|^{-1}$, that is with precision which is enough to find the amplitude of the minimal quantum fluctuations.

As the universe expands, the absolute value $|\eta|$ decreases, and so for a given k the value of $k|\eta|$ eventually becomes smaller than unity. It is clear from (7.15) that the physical scale of the mode with a given k becomes of order the curvature scale H_Λ^{-1} at $\eta = \eta_k$ when $k|\eta_k| \sim 1$. We call this time the moment of (event) horizon crossing and refer to the modes with $k|\eta| \gg 1$ and $k|\eta| \ll 1$ as the *subhorizon* and *superhorizon* modes respectively. The subhorizon modes are eventually stretched by expansion and start to feel the curvature of the universe. After horizon crossing, for $k|\eta| \ll 1$, we can neglect the k^2 term in (7.13) and hence

$$v_k'' - \left(2 - \frac{m^2}{H_\Lambda^2}\right)\frac{1}{\eta^2} v_k = 0.$$

The general solution of this equation is

$$v_k(\eta) = A_k |\eta|^{n_1} + B_k |\eta|^{n_2}, \tag{7.17}$$

where

$$n_{1,2} \equiv \frac{1}{2} \pm \sqrt{\frac{9}{4} - \frac{m^2}{H_\Lambda^2}}.$$

At late times ($\eta \to 0$) the term proportional to $B|\eta|^{n_2}$ dominates.

7.2.1 Bunch–Davies vacuum

The superhorizon modes do not oscillate and hence the notion of a particle is not well-defined for $k \ll |\eta|^{-1}$. Moreover, for $m^2 < 2H_\Lambda^2$ the effective mass squared,

$$m_{\text{eff}}^2(\eta) = -\left(2 - \frac{m^2}{H_\Lambda^2}\right)\frac{1}{\eta^2},$$

is negative and the lowest energy state does not exist for the superhorizon modes. However, there exists a preferred quantum state called the Bunch–Davies vacuum. This state is de Sitter invariant and does not change with time. Let us construct the mode functions for the Bunch–Davies vacuum. Considering a mode with a given comoving k, we find that in the far remote past ($\eta \to -\infty$), when $k|\eta| \gg 1$, this mode does not feel the curvature and one can fix the initial conditions by requiring that

$$v_k(\eta) \to \frac{1}{\sqrt{\omega_k}} e^{i\omega_k \eta} \tag{7.18}$$

as $\eta \to -\infty$. In other words, we select the minimal excitation state in the remote past. Assuming that $m < \frac{3}{2} H\Lambda$ and using the results of Exercise 7.1, we find that the functions

$$v_k(\eta) = \sqrt{\frac{\pi |\eta|}{2}} \left[J_n(k|\eta|) - iY_n(k|\eta|) \right], \quad n \equiv \sqrt{\frac{9}{4} - \frac{m^2}{H^2}}, \tag{7.19}$$

have the required asymptotic (7.18). These mode functions determine the *Bunch–Davies vacuum* $|0_{BD}\rangle$ in the standard way: $|0_{BD}\rangle$ is annihilated by all operators $\hat{a}^-_{\mathbf{k}}$ entering expansion (6.20) with v_k given in (7.19). To verify that $|0_{BD}\rangle$ actually describes a time-independent state, let us find how the amplitude of fluctuations of the original field ϕ depends on the physical wavenumber $k_{ph} = k/a$. The field ϕ is related to the auxiliary field by $\phi = a^{-1}\chi$. Taking into account that $k|\eta| = k_{ph} H_\Lambda^{-1}$, we find (see (6.52))

$$\begin{aligned} \delta_\phi(k_{ph}) &= \frac{1}{2\pi} a^{-1} k^{3/2} |v_k(\eta)| \\ &= \frac{H_\Lambda}{\sqrt{8\pi}} \left(\frac{k_{ph}}{H_\Lambda}\right)^{3/2} \left[J_n^2\left(\frac{k_{ph}}{H_\Lambda}\right) + Y_n^2\left(\frac{k_{ph}}{H_\Lambda}\right) \right]^{1/2}, \end{aligned} \tag{7.20}$$

and hence the amplitude of the fluctuations on a given physical scale does not depend on time. Using the asymptotics of the Bessel functions, one obtains from (7.20) that

$$\delta_\phi \simeq \begin{cases} \dfrac{k_{ph}}{2\pi}, & k_{ph} \gg H_\Lambda, \\ \dfrac{2^n \Gamma(n)}{\sqrt{8\pi^3}} H_\Lambda \left(\dfrac{k_{ph}}{H_\Lambda}\right)^{\frac{3}{2}-n}, & k_{ph} \ll H_\Lambda. \end{cases} \tag{7.21}$$

In Fig. 7.2 we show the amplitude of quantum fluctuations as a function of the physical wavelength $L_{ph} = 2\pi/k_{ph}$. For short-wavelength modes, the Bunch–Davies spectrum is in agreement with the spectrum of fluctuations in Minkowski space (see (6.53)). This confirms our naive expectations that the curvature is not very relevant on subcurvature scales. When $m^2 \ll H_\Lambda^2$, we have

$$\delta_\phi \propto L_{ph}^{-m^2/3H_\Lambda^2} \tag{7.22}$$

for $L_{ph} \gg H_\Lambda^{-1}$ and the amplitude of the fluctuations decays only weakly with the scale. In the case of a massless field, this amplitude becomes scale independent on supercurvature scales.

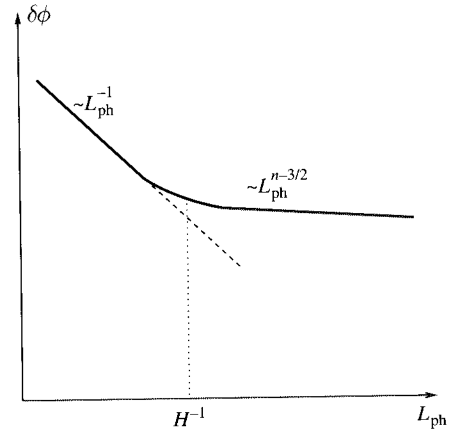

Fig. 7.2 The fluctuation amplitude $\delta\phi_{L_{\mathrm{ph}}}(\eta)$ as function of L_{ph} at fixed time η. The dashed line shows the amplitude of fluctuations in Minkowski spacetime. (Logarithmic scaling is used for both axes.)

7.3 Fluctuations in inflationary universe

The de Sitter universe plays an important role in cosmology. It can be used as a good approximation for the stage of accelerated expansion, known as *inflation*. The inflationary stage has a finite duration and therefore we need only a "piece" of the entire de Sitter spacetime to describe it. Let us assume that inflation begins at time $\eta = \eta_i$ and is over by $\eta = \eta_f$. Within the time interval $\eta_i < \eta < \eta_f$, we approximate the expansion by a flat de Sitter solution. In this case the quantum state of fields at the beginning of inflation depends on the previous history of the universe and can be very different from the Bunch–Davies state. Let us show that regardless of the initial conditions, the spectrum of fluctuations converges to the Bunch–Davies spectrum as the universe expands. To simplify the calculations we assume that at $\eta = \eta_i$ the subhorizon modes are in the state of minimal excitation. This means that for the modes with $k\,|\eta_i| \gg 1$,

$$v_k(\eta) \approx \frac{1}{\sqrt{k}} e^{ik\eta} \tag{7.23}$$

for $\eta_i < \eta < -1/k$ (recall that η is negative). At $\eta = \eta_i$ the minimal excitation state cannot be defined for the modes with $k\,|\eta_i| < 1$ and their spectrum is entirely determined by the unknown preinflationary evolution. Let us see what happens to a subcurvature mode when it crosses the horizon at time $\eta_k \simeq -1/k$. For $\eta > \eta_k$ the asymptotic solution for $v_k(\eta)$ is given in (7.17) where the first term eventually becomes negligible. Ignoring the numerical coefficients of order unity and matching solutions (7.23) and (7.17) at $\eta_k \simeq -1/k$, we obtain

$$v_k(\eta) \sim \frac{1}{\sqrt{k}} |k\eta|^{\frac{1}{2}-n} \tag{7.24}$$

for $\eta > \eta_k$. Thus, after the beginning of inflation we have

$$\delta_\phi\left(L_{\text{ph}}, \eta\right) = \frac{k^{3/2}\left|v_k(\eta)\right|}{2\pi a(\eta)} \sim \begin{cases} L_{\text{ph}}^{-1}, & L_{\text{ph}} < H_\Lambda^{-1}, \\ H_\Lambda \left[\dfrac{L_{\text{ph}}}{H_\Lambda^{-1}}\right]^{n-\frac{3}{2}}, & H_\Lambda^{-1}\dfrac{\eta_i}{\eta} > L_{\text{ph}} > H_\Lambda^{-1}, \\ \text{unknown}, & L_{\text{ph}} > H_\Lambda^{-1}\dfrac{\eta_i}{\eta}. \end{cases} \tag{7.25}$$

The evolution of the spectrum with time is shown in Fig. 7.3. We see that the perturbations stretched from subhorizon scales build the longwave part of the Bunch–Davies spectrum. The unknown part of the spectrum is redshifted by the expansion to very large physical scales

$$L_{\text{ph}} > H_\Lambda^{-1}\left(\eta_i/\eta\right) = H_\Lambda^{-1} \exp\left(H_\Lambda\left(t-t_i\right)\right).$$

If inflation continued forever ($\eta \to -0$), then an arbitrary initial state would evolve into the Bunch–Davies vacuum. However, because the duration of inflation is finite, the Bunch–Davies spectrum is formed only on the scales $L_{\text{ph}} < H_\Lambda^{-1}\eta_f/\eta$.

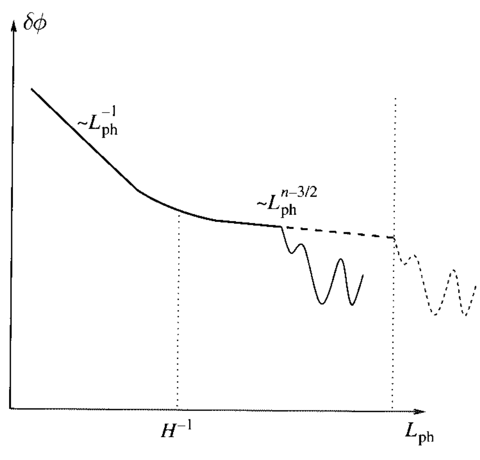

Fig. 7.3 The fluctuation spectrum at time $\eta = \eta_1$ (solid curve) and at later time $\eta = \eta_2$ (dashed curve). The wiggly lines in the infrared region of the spectrum correspond to the scales where the fluctuation amplitude is unknown; this region moves towards very large scales.

In realistic models, the Hubble parameter $H \equiv \dot{a}/a$ changes slightly and decreases towards the end of inflation. Let us find how this influences the spectrum of the generated fluctuations for a massless scalar field. We have found that in this case the amplitude of the large-scale fluctuations must be scale-independent or, in other words, the spectrum is flat, if H is strictly constant. It is clear that the change in H will cause deviations from the flat spectrum. For a massless scalar field ($m = 0$) the mode equation takes the form

$$v_k'' + \left(k^2 - \frac{a''}{a}\right) v_k = 0. \tag{7.26}$$

The effective mass squared is expressed in terms of the Hubble parameter H as

$$m_{\text{eff}}^2 \equiv -\frac{a''}{a} = -a^2\left(2H^2 + \dot{H}\right),$$

where the dot denotes the derivative with respect to the physical time t. During inflation, the Hubble parameter changes insignificantly during a Hubble time H^{-1} and hence $|\dot{H}| \ll H^2$. Given a mode with a comoving wavenumber $k \gg aH = \dot{a}$, we find that this mode which had originally a physical wavelength $L_{\text{ph}} < H^{-1}$ is stretched to the curvature scale at the moment t_k determined by the condition $k \simeq a_k H_k$. Later on, at $t > t_k$, its physical wavelength exceeds the curvature scale. Note that subcurvature scales can be stretched to supercurvature scales only during inflation when the expansion is accelerating ($\dot{a}$ grows). In a decelerating universe, $\dot{a}$ decreases and the condition $k \gg aH = \dot{a}$ holds at all times if it is satisfied initially. Thus in a decelerating Friedmann universe the subcurvature modes will never feel the effects of curvature.

Considering a subcurvature mode with $k \gg a_i H_i$, we find that for $t_i < t < t_k$ the mode function corresponding to the minimal excitation state is

$$v_k \approx \frac{1}{\sqrt{k}} e^{ik\eta}. \tag{7.27}$$

At $t = t_k$ this mode leaves the (event) horizon and for $t > t_k$ we can neglect the k^2 term in equation (7.26) which then becomes

$$v_k'' - \frac{a''}{a} v_k \simeq 0.$$

The general solution of this equation is

$$v_k = A_k a + B_k a \int \frac{d\eta}{a^2}. \tag{7.28}$$

One can verify that the first term (proportional to a) becomes dominant at late times. Therefore, neglecting the second mode in (7.28) and matching solutions (7.27) and (7.28) at $t = t_k$ (by order of magnitude), we obtain

$$v_k \simeq \frac{1}{\sqrt{k}} \frac{a(t)}{a_k} \simeq \frac{H_k}{k^{3/2}} a(t), \tag{7.29}$$

where a_k and H_k are the values of the scale factor and the Hubble parameter at the moment of horizon crossing determined by the condition $k \simeq a_k H_k$. The spectrum of the fluctuations at $t \gg t_i$ is then

$$\delta_\phi \sim \frac{k^{3/2} |v_k|}{a(t)} \sim H_k \quad \text{for} \quad H^{-1}(t) < L_{\text{ph}} < H^{-1} \left(\frac{a(t)}{a(t_i)} \right). \tag{7.30}$$

Because the Hubble parameter is decreasing during inflation, the value of H_k is larger for those modes which left the horizon earlier. As a result, the amplitude of fluctuations is slightly higher toward the large scales and the resulting fluctuation spectrum is red-tilted within the corresponding range of scales (see Fig. 7.4).

These results can be directly applied to derive the spectrum of long-wavelength gravitational waves produced during inflation. One can show that the quantization of gravitational waves in an expanding universe can be reduced to the problem of quantization of a massless scalar field. Therefore the spectrum of gravitational waves produced in an accelerated universe also deviates from the flat spectrum. Since the Hubble parameter changes very slowly, the amplitude of fluctuations depends on the scale only logarithmically. The observed structure of the universe

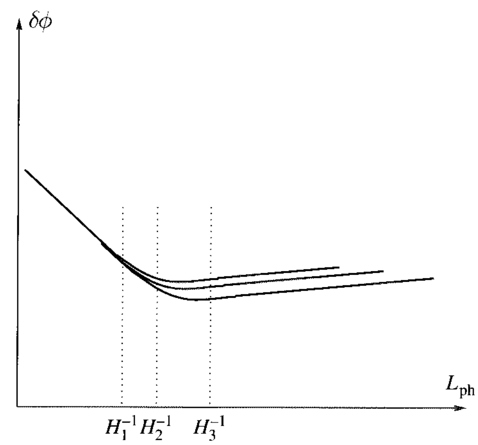

Fig. 7.4 The fluctuation spectrum resulting from inflation at three consecutive moments of time where the Hubble parameter has values H_1^{-1}, H_2^{-1}, H_3^{-1}. The spectrum is red-tilted towards large scales in the region $L_{\text{ph}} > H^{-1}$.

can be explained by considering the quantum scalar metric perturbations during inflation. This is a technically much more involved problem. However, the physical explanation of the production of the primordial inhomogeneities with a nearly scale-invariant spectrum is not very different from that presented above. In the case of matter inhomogeneities the "backreaction" of the gravitational field potential makes the m^2 term in (7.22) negative and this adds to the red tilt of the spectrum of scalar metric perturbations when compared to the spectrum of gravity waves.[1]

[1] For a detailed treatment of the quantum theory of cosmological perturbations, we refer the reader to the book *Physical Foundations of Cosmology* by V. Mukhanov (Cambridge University Press, 2005).