6

Quantum fields in expanding universe

Summary Quantization of a scalar field in a Friedmann universe. Bogolyubov transformations. Choice of the vacuum state. Particle production. Correlation functions. Amplitude of quantum fluctuations.

Scalar field quantization is tremendously simplified in an isotropic and homogeneous expanding universe. Homogeneity and isotropy dictate a preferable choice of the spacetime foliation and a preferable time parametrization. In this chapter we will consider a minimally coupled quantum scalar field and study how its state changes in a homogeneous and isotropic gravitational background.

6.1 Classical scalar field in expanding background

For simplicity, we consider only the case of the spatially flat Friedmann universe. In the preferred coordinate system where the symmetries of the spacetime are manifest, the metric takes the form

$$ds^2 = dt^2 - a^2(t)\delta_{ik}dx^i dx^k. \tag{6.1}$$

Note that although the three-dimensional surfaces of constant time are flat, the spacetime is nevertheless curved. It is convenient to introduce the *conformal time*

$$\eta(t) \equiv \int^t \frac{dt}{a(t)},$$

instead of the physical time t. With this new coordinate interval (6.1) is

$$ds^2 = a^2(\eta)\left[d\eta^2 - \delta_{ik}dx^i dx^k\right] = a^2(\eta)\eta_{\mu\nu}dx^\mu dx^\nu, \tag{6.2}$$

and it is obvious that the metric is conformally equivalent to the Minkowski metric $\eta_{\mu\nu}$.

A real minimally coupled massive scalar field $\phi(x)$ in a curved spacetime is described by the action

$$S = \frac{1}{2}\int \sqrt{-g}d^4x \left[g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} - m^2\phi^2\right]. \tag{6.3}$$

With the substitution $g^{\alpha\beta} = a^{-2}\eta^{\alpha\beta}$ and $\sqrt{-g} = a^4$ this action becomes

$$S = \frac{1}{2}\int d^3\mathbf{x}\, d\eta\, a^2 \left[\phi'^2 - (\nabla\phi)^2 - m^2a^2\phi^2\right], \tag{6.4}$$

where the prime $'$ denotes derivatives with respect to the conformal time η. Moreover, introducing the auxiliary field

$$\chi \equiv a(\eta)\phi, \tag{6.5}$$

we can rewrite action (6.4) in terms of χ as

$$S = \frac{1}{2}\int d^3\mathbf{x}\, d\eta \left[\chi'^2 - (\nabla\chi)^2 - \left(m^2a^2 - \frac{a''}{a}\right)\chi^2\right], \tag{6.6}$$

where the total derivative terms were omitted.

Exercise 6.1
Derive (6.6) from (6.4).

Variation of the action (6.6) with respect to χ gives the following equation of motion,

$$\chi'' - \Delta\chi + \left(m^2a^2 - \frac{a''}{a}\right)\chi = 0. \tag{6.7}$$

Comparing (6.7) with (4.7), we find that the field χ obeys the same equation of motion as a massive scalar field in Minkowski spacetime, except that the *effective mass*,

$$m_{\text{eff}}^2(\eta) \equiv m^2a^2 - \frac{a''}{a}, \tag{6.8}$$

becomes time-dependent. This time dependence of the effective mass accounts for the interaction of the scalar field with the gravitational background.

Thus, the problem of quantization of a scalar field ϕ in a flat Friedmann universe is reduced to the mathematically equivalent problem of quantization of a free scalar field χ in Minkowski spacetime. All information about the influence of the gravitational field on ϕ is encapsulated in the time-dependent mass $m_{\text{eff}}(\eta)$. Note that action (6.6) is explicitly time-dependent, so the energy of the scalar field χ is not conserved. In quantum theory this leads to particle creation; the energy for new particles is supplied by the classical gravitational field.

6.1.1 Mode expansion

Expanding the field χ in Fourier modes,

$$\chi(\mathbf{x}, \eta) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \chi_{\mathbf{k}}(\eta) e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{6.9}$$

and substituting this expansion into (6.7) we find that the Fourier modes $\chi_{\mathbf{k}}(\eta)$ satisfy a set of decoupled ordinary differential equations

$$\chi_{\mathbf{k}}'' + \omega_k^2(\eta)\chi_{\mathbf{k}} = 0, \tag{6.10}$$

where

$$\omega_k^2(\eta) \equiv k^2 + m_{\text{eff}}^2(\eta) = k^2 + m^2 a^2(\eta) - \frac{a''}{a}. \tag{6.11}$$

Because the field $\chi(\mathbf{x}, \eta)$ is real, i.e. $\chi^*(\mathbf{x}, \eta) = \chi(\mathbf{x}, \eta)$, the complex Fourier modes $\chi_{\mathbf{k}}(\eta)$ must satisfy the condition

$$\chi_{\mathbf{k}}^*(\eta) = \chi_{-\mathbf{k}}(\eta). \tag{6.12}$$

Since $\omega_k^2(\eta)$ in equation (6.10) depends only on $k \equiv |\mathbf{k}|$, its general solution can be written as

$$\chi_{\mathbf{k}}(\eta) = \frac{1}{\sqrt{2}}\left[a_{\mathbf{k}}^- v_k^*(\eta) + a_{-\mathbf{k}}^+ v_k(\eta)\right]. \tag{6.13}$$

Here $v_k(\eta)$ and its complex conjugate $v_k^*(\eta)$ are two linearly independent solutions of (6.10) which are the same for all Fourier modes with the given magnitude of the wavevector $k \equiv |\mathbf{k}|$, and $a_{\mathbf{k}}^{\pm}$ are two complex constants of integration that can depend also on the direction of vector $\mathbf{k}$. The index $-\mathbf{k}$ in the second term and the factor $1/\sqrt{2}$ are chosen for later convenience. Since χ is real, it follows from (6.12) that $a_{\mathbf{k}}^+ = \left(a_{\mathbf{k}}^-\right)^*$.

Exercise 6.2
Verify that if v_k satisfies (6.10), then the *Wronskian*

$$W\left[v_k, v_k^*\right] \equiv v_k' v_k^* - v_k v_k^{*\prime} = 2i\text{Im}\left(v' v^*\right) \tag{6.14}$$

is time-independent. Show that $W\left[v_k, v_k^*\right] \neq 0$ if and only if v_k and v_k^* are linearly independent solutions. Verify that the coefficients $a_{\mathbf{k}}^{\pm}$ can be expressed in terms of $\chi_{\mathbf{k}}(\eta)$ and $v_k(\eta)$ as:

$$a_{\mathbf{k}}^- = \sqrt{2}\frac{v_k'\chi_{\mathbf{k}} - v_k\chi_{\mathbf{k}}'}{v_k' v_k^* - v_k v_k^{*\prime}} = \sqrt{2}\frac{W\left[v_k, \chi_{\mathbf{k}}\right]}{W\left[v_k, v_k^*\right]}; \quad a_{\mathbf{k}}^+ = \left(a_{\mathbf{k}}^-\right)^*. \tag{6.15}$$

Note that the numerators and denominators in (6.15) are time-independent.

When $v(\eta)$ is multiplied by a constant, $v(\eta) \to \lambda v(\eta)$, the Wronskian $W\left[v, v^*\right]$ scales as $|\lambda|^2$. Therefore if $W \neq 0$ we can always normalize v_k in such a way that

$\text{Im}(\dot{v}v^*) = 1$. In this case a complex solution $v_k(\eta)$ is called a *mode function*. It follows from the results of Exercise 6.2 that $v_k(\eta)$ and $v_k^*(\eta)$ are linearly independent.

Substituting (6.13) into (6.9) we find

$$\chi(\mathbf{x},\eta) = \frac{1}{\sqrt{2}}\int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\left[a_\mathbf{k}^- v_k^*(\eta) + a_{-\mathbf{k}}^+ v_k(\eta)\right]e^{i\mathbf{k}\cdot\mathbf{x}}$$
$$= \frac{1}{\sqrt{2}}\int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\left[a_\mathbf{k}^- v_k^*(\eta)e^{i\mathbf{k}\cdot\mathbf{x}} + a_\mathbf{k}^+ v_k(\eta)e^{-i\mathbf{k}\cdot\mathbf{x}}\right], \quad (6.16)$$

where in the second term the integration variable $\mathbf{k}$ was changed from $\mathbf{k}$ to $-\mathbf{k}$ to make the integrand a manifestly real expression.

Remark: isotropic mode functions In equation (6.13) we expressed all Fourier modes $\chi_\mathbf{k}(\eta)$ with a given $|\mathbf{k}| = k$ in terms of the *same* mode function $v_k(\eta)$. This *isotropic* choice of the mode functions is possible because of the isotropy of the Friedmann universe where ω_k depends only on $k = |\mathbf{k}|$.

6.2 Quantization

The field χ is quantized by imposing the standard equal-time commutation relations on the field operator $\hat{\chi}$ and its canonically conjugate momentum $\hat{\pi} \equiv \hat{\chi}'$,

$$[\hat{\chi}(\mathbf{x},\eta), \hat{\pi}(\mathbf{y},\eta)] = i\delta(\mathbf{x}-\mathbf{y}); \quad (6.17)$$
$$[\hat{\chi}(\mathbf{x},t), \hat{\chi}(\mathbf{y},t)] = [\hat{\pi}(\mathbf{x},t), \hat{\pi}(\mathbf{y},t)] = 0. \quad (6.18)$$

The Hamiltonian for the quantum field $\hat{\chi}$ is

$$\hat{H}(\eta) = \frac{1}{2}\int d^3\mathbf{x}\left[\hat{\pi}^2 + (\nabla\hat{\chi})^2 + m_{\text{eff}}^2(\eta)\hat{\chi}^2\right]. \quad (6.19)$$

The creation and annihilation operators $\hat{a}_\mathbf{k}^\pm$ can be introduced via the mode operators $\hat{\chi}_\mathbf{k}$ as in Chapter 4. However, a quicker way is to begin directly with the mode expansion (6.16) considering the constants of integration $a_\mathbf{k}^\pm$ as operators $\hat{a}_\mathbf{k}^\pm$. Then the field operator $\hat{\chi}$ is expanded as

$$\hat{\chi}(\mathbf{x},\eta) = \frac{1}{\sqrt{2}}\int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\left(e^{i\mathbf{k}\cdot\mathbf{x}}v_k^*(\eta)\hat{a}_\mathbf{k}^- + e^{-i\mathbf{k}\cdot\mathbf{x}}v_k(\eta)\hat{a}_\mathbf{k}^+\right), \quad (6.20)$$

where the mode functions $v_k(\eta)$ obey the equations

$$v_k'' + \omega_k^2(\eta)v_k = 0, \quad \omega_k(\eta) \equiv \sqrt{k^2 + m_{\text{eff}}^2(\eta)}, \quad (6.21)$$

and satisfy the normalization condition

$$\text{Im}\left(v_k' v_k^*\right) = \frac{v_k' v_k^* - v_k v_k^{*\prime}}{2i} = 1. \quad (6.22)$$

Substituting (6.20) into (6.17) and taking into account (6.22) we find that the operators $\hat{a}^{\pm}_{\mathbf{k}}$ satisfy the commutation relations

$$[\hat{a}^{-}_{\mathbf{k}}, \hat{a}^{+}_{\mathbf{k}'}] = \delta(\mathbf{k}-\mathbf{k}'), \quad [\hat{a}^{-}_{\mathbf{k}}, \hat{a}^{-}_{\mathbf{k}'}] = [\hat{a}^{+}_{\mathbf{k}}, \hat{a}^{+}_{\mathbf{k}'}] = 0, \tag{6.23}$$

and thus can be interpreted as the creation and annihilation operators.

Exercise 6.3
Verify this last statement.

Remark: complex scalar field If χ were a complex field, then in general $\chi^*_{\mathbf{k}} \neq \chi_{-\mathbf{k}}$ and the mode expansion would be written as

$$\hat{\chi}(\mathbf{x}, \eta) = \frac{1}{\sqrt{2}} \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \left(e^{i\mathbf{k}\cdot\mathbf{x}} v^*_k(\eta)\hat{a}^-_{\mathbf{k}} + e^{-i\mathbf{k}\cdot\mathbf{x}} v_k(\eta)\hat{b}^+_{\mathbf{k}} \right),$$

where the operators $\hat{a}^-_{\mathbf{k}}$ and $\hat{b}^+_{\mathbf{k}}$ are independent. In this case we have two sets of creation and annihilation operators, $\hat{a}^{\pm}_{\mathbf{k}}$ and $\hat{b}^{\pm}_{\mathbf{k}}$, satisfying $(\hat{a}^-_{\mathbf{k}})^\dagger = \hat{a}^+_{\mathbf{k}}$ and $(\hat{b}^-_{\mathbf{k}})^\dagger = \hat{b}^+_{\mathbf{k}}$. The operators $\hat{a}^+_{\mathbf{k}}$ and $\hat{b}^+_{\mathbf{k}}$ create the particles and antiparticles respectively. This agrees with the picture that a complex scalar field describes particles which are different from their antiparticles. For a real scalar field particles are their own antiparticles.

6.3 Bogolyubov transformations

The operators $\hat{a}^{\pm}_{\mathbf{k}}$ can be used to construct the basis of quantum states in the Hilbert space. However, the corresponding states acquire an unambiguous physical interpretation only after the particular mode functions $v_k(\eta)$ are selected. The normalization condition (6.22) is not enough to completely specify the complex solutions $v_k(\eta)$ of the second-order differential equation (6.21).

In fact, the functions

$$u_k(\eta) = \alpha_k v_k(\eta) + \beta_k v^*_k(\eta), \tag{6.24}$$

where α_k and β_k are time-independent complex coefficients, also satisfy equation (6.21). Moreover, if the coefficients α_k and β_k obey the condition

$$|\alpha_k|^2 - |\beta_k|^2 = 1, \tag{6.25}$$

then the functions $u_k(\eta)$ satisfy the normalization condition (6.22) and therefore they can be used as the mode functions instead of $v_k(\eta)$.

Exercise 6.4
Verify that if (6.25) is satisfied then $\mathrm{Im}(u'_k u^*_k) = 1$.

In terms of the mode functions $u_k(\eta)$ the field operator expansion takes the following form,

$$\hat{\chi}(\mathbf{x}, \eta) = \frac{1}{\sqrt{2}} \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \left[e^{i\mathbf{k}\cdot\mathbf{x}} u^*_k(\eta)\hat{b}^-_{\mathbf{k}} + e^{-i\mathbf{k}\cdot\mathbf{x}} u_k(\eta)\hat{b}^+_{\mathbf{k}} \right], \tag{6.26}$$

where $\hat{b}_{\mathbf{k}}^{\pm}$ is another set of the creation and annihilation operators satisfying the standard commutation relations (6.23). To determine how the operators $\hat{b}_{\mathbf{k}}^{\pm}$ are related to $\hat{a}_{\mathbf{k}}^{\pm}$, we note that the two expressions (6.20) and (6.26) for the same field operator $\hat{\chi}(\mathbf{x}, \eta)$ agree only if

$$e^{i\mathbf{k}\cdot\mathbf{x}}\left[u_k^*(\eta)\hat{b}_{\mathbf{k}}^- + u_k(\eta)\hat{b}_{-\mathbf{k}}^+\right] = e^{i\mathbf{k}\cdot\mathbf{x}}\left[v_k^*(\eta)\hat{a}_{\mathbf{k}}^- + v_k(\eta)\hat{a}_{-\mathbf{k}}^+\right].$$

Substituting here the expression for u_k in terms of v_k from (6.24), we find the following relation between the operators $\hat{a}_{\mathbf{k}}^{\pm}$ and $\hat{b}_{\mathbf{k}}^{\pm}$:

$$\hat{a}_{\mathbf{k}}^- = \alpha_k^*\hat{b}_{\mathbf{k}}^- + \beta_k\hat{b}_{-\mathbf{k}}^+, \quad \hat{a}_{\mathbf{k}}^+ = \alpha_k\hat{b}_{\mathbf{k}}^+ + \beta_k^*\hat{b}_{-\mathbf{k}}^-. \tag{6.27}$$

The above relations are called the *Bogolyubov transformation.* One can reverse (6.27) to obtain

$$\hat{b}_{\mathbf{k}}^- = \alpha_k\hat{a}_{\mathbf{k}}^- - \beta_k\hat{a}_{-\mathbf{k}}^+, \quad \hat{b}_{\mathbf{k}}^+ = \alpha_k^*\hat{a}_{\mathbf{k}}^+ - \beta_k^*\hat{a}_{-\mathbf{k}}^-. \tag{6.28}$$

The *Bogolyubov coefficients* α_k and β_k can in turn be expressed in terms of the mode functions $v_k(\eta)$ and $u_k(\eta)$. From the relations

$$u_k(\eta) = \alpha_k v_k(\eta) + \beta_k v_k^*(\eta),$$
$$u_k'(\eta) = \alpha_k v_k'(\eta) + \beta_k v_k^{*\prime}(\eta),$$

we find

$$\alpha_k = \frac{W(u_k, v_k^*)}{2i}, \quad \beta_k = \frac{W(v_k, u_k)}{2i}, \tag{6.29}$$

where W is the Wronskian.

6.4 Hilbert space; "*a*- and *b*-particles"

Both sets of the operators $\hat{a}_{\mathbf{k}}^{\pm}$ and $\hat{b}_{\mathbf{k}}^{\pm}$ can be used to build orthonormal bases in the Hilbert space. The two different "vacuum vectors" $|_{(a)}0\rangle$ and $|_{(b)}0\rangle$ can be defined in the standard way,

$$\hat{a}_{\mathbf{k}}^-\,|_{(a)}0\rangle = 0, \quad \hat{b}_{\mathbf{k}}^-\,|_{(b)}0\rangle = 0,$$

for all $\mathbf{k}$. We call them the "a-vacuum" and the "b-vacuum" respectively. Two sets of excited states,

$$\left|_{(a)}m_{\mathbf{k}_1}, n_{\mathbf{k}_2}, \ldots\right\rangle \equiv \frac{1}{\sqrt{m!n!\ldots}}\left[\left(\hat{a}_{\mathbf{k}_1}^+\right)^m\left(\hat{a}_{\mathbf{k}_2}^+\right)^n\ldots\right]|_{(a)}0\rangle \tag{6.30}$$

and

$$\left|_{(b)}m_{\mathbf{k}_1}, n_{\mathbf{k}_2}, \ldots\right\rangle \equiv \frac{1}{\sqrt{m!n!\ldots}}\left[\left(\hat{b}_{\mathbf{k}_1}^+\right)^m\left(\hat{b}_{\mathbf{k}_2}^+\right)^n\ldots\right]|_{(b)}0\rangle, \tag{6.31}$$

describe the "a- and b-particles" respectively. An arbitrary quantum state $|\psi\rangle$ can be written as a linear combination of the excited states,

$$|\psi\rangle = \sum_{m,n,\dots} C^{(a)}_{mn\dots} \left|{}_{(a)}m_{\mathbf{k}_1}, n_{\mathbf{k}_2}, \dots\right\rangle = \sum_{m,n,\dots} C^{(b)}_{mn\dots} \left|{}_{(b)}m_{\mathbf{k}_1}, n_{\mathbf{k}_2}, \dots\right\rangle,$$

and the probability to find m "a-particles" in the mode $\mathbf{k}_1$, etc. is $\left|C^{(a)}_{mn\dots}\right|^2$. For the "$b$-particles" the corresponding probabilities are given by $\left|C^{(b)}_{mn\dots}\right|^2$.

The b-states are in general different from the a-states. In particular, if $\beta_k \neq 0$ then the "b-vacuum" contains "a-particles." To verify this let us calculate the expectation value of the a-particle number operator $\hat{N}^{(a)}_{\mathbf{k}} = \hat{a}^+_{\mathbf{k}}\hat{a}^-_{\mathbf{k}}$ in the state $\left|{}_{(b)}0\right\rangle$. Using (6.27) we obtain

$$\begin{aligned}\left\langle{}_{(b)}0\right| \hat{N}^{(a)}_{\mathbf{k}} \left|{}_{(b)}0\right\rangle &= \left\langle{}_{(b)}0\right| \hat{a}^+_{\mathbf{k}}\hat{a}^-_{\mathbf{k}} \left|{}_{(b)}0\right\rangle \\ &= \left\langle{}_{(b)}0\right| \left(\alpha_k \hat{b}^+_{\mathbf{k}} + \beta^*_k \hat{b}^-_{-\mathbf{k}}\right)\left(\alpha^*_k \hat{b}^-_{\mathbf{k}} + \beta_k \hat{b}^+_{-\mathbf{k}}\right) \left|{}_{(b)}0\right\rangle \\ &= \left\langle{}_{(b)}0\right| \left(\beta^*_k \hat{b}^-_{-\mathbf{k}}\right)\left(\beta_k \hat{b}^+_{-\mathbf{k}}\right) \left|{}_{(b)}0\right\rangle = |\beta_k|^2\,\delta^{(3)}(0). \end{aligned} \tag{6.32}$$

The divergent factor $\delta^{(3)}(0)$ accounts for an infinite spatial volume and hence the mean *density* of the a-particles in the mode $\mathbf{k}$ is

$$n_{\mathbf{k}} = |\beta_k|^2 .$$

The total mean density of all particles,

$$n = \int d^3\mathbf{k}\, |\beta_k|^2 ,$$

is finite only if $|\beta_k|^2$ decays faster than k^{-3} for large k.

The "b-vacuum" can be expressed as a superposition of excited a-particle states as (see Exercise 6.5)

$$\begin{aligned}\left|{}_{(b)}0\right\rangle &= \left[\prod_{\mathbf{k}} \frac{1}{|\alpha_k|^{1/2}} \exp\left(\frac{\beta_k}{2\alpha_k}\hat{a}^+_{\mathbf{k}}\hat{a}^+_{-\mathbf{k}}\right)\right]\left|{}_{(a)}0\right\rangle \\ &= \prod_{\mathbf{k}} \frac{1}{|\alpha_k|^{1/2}} \left(\sum_{n=0}^{\infty}\left(\frac{\beta_k}{2\alpha_k}\right)^n \left|{}_{(a)}n_{\mathbf{k}}, n_{-\mathbf{k}}\right\rangle\right). \end{aligned} \tag{6.33}$$

Because of the isotropy the particles come in pairs with opposite momenta $\mathbf{k}$ and $-\mathbf{k}$.

Exercise 6.5
Derive (6.33).

Quantum states defined by an exponential of a quadratic combination of creation operators acting on the vacuum, as in (6.33), are called *squeezed* states. The "b-vacuum" is therefore a squeezed state with respect to the "a-vacuum." Similarly, the "a-vacuum" is a squeezed state with respect to the "b-vacuum."

The "b-vacuum" is normalized by the infinite product $\prod_{\mathbf{k}} |\alpha_k|$. This product converges only if $|\alpha_k|$ tends to unity rapidly enough as $k \to \infty$ or, more precisely, if $|\beta_k|^2$ vanishes faster than k^{-3} for large k. Otherwise, the vacuum state $\left|_{(b)}0\right\rangle$ is not expressible as a normalized combination of a-states, and the Bogolyubov transformation is not well-defined. Note that the same condition guarantees that the total mean number density is finite.

6.5 Choice of the physical vacuum

It is clear from the above consideration that the particle interpretation of the theory depends on the choice of the mode functions. We have seen that the a-vacuum $\left|_{(a)}0\right\rangle$, being a state without a-particles, nevertheless contains b-particles. A natural question is whether the a- or b-particles correspond to the observable particles. So far, all mode functions related by linear transformations (6.24) are on the same footing and our problem here is to determine the preferable set of the mode functions that describe the "actual" physical vacuum and particles.

6.5.1 The instantaneous lowest-energy state

In Chapter 4 the vacuum was defined as an eigenstate of the Hamiltonian with the lowest possible energy. This allowed us to choose the preferable set of mode functions and thus to unambiguously determine the physical vacuum. However, in the case under consideration the Hamiltonian (6.19) depends explicitly on time and thus does not possess time-independent eigenvectors that could serve as a vacuum. Nevertheless, given a particular moment of time η_0 we can still define the *instantaneous* vacuum $\left|_{\eta_0}0\right\rangle$ as the lowest-energy state of the Hamiltonian $\hat{H}(\eta_0)$.

To find a set of mode functions that determine $\left|_{\eta_0}0\right\rangle$, we first compute the expectation value $\left\langle_{(v)}0\right| \hat{H}(\eta_0) \left|_{(v)}0\right\rangle$ for the "vacuum" state $\left|_{(v)}0\right\rangle$ determined by arbitrarily chosen mode functions $v_k(\eta)$. Then we shall minimize this expectation value with respect to $v_k(\eta)$. (A standard result of the linear algebra is that the minimization of $\langle x| \hat{A} |x\rangle$ with respect to normalized vectors $|x\rangle$ is equivalent to finding the eigenvector $|x\rangle$ of the operator $\hat{A}$ with the smallest eigenvalue.)

Taking into account that $\hat{\pi} \equiv \hat{\chi}'$ and substituting the mode expansion (6.20) into (6.19) we find:

$$\hat{H}(\eta) = \frac{1}{4}\int d^3\mathbf{k}\left[\hat{a}^-_{\mathbf{k}}\hat{a}^-_{-\mathbf{k}}F^*_k + \hat{a}^+_{\mathbf{k}}\hat{a}^+_{-\mathbf{k}}F_k + \left(2\hat{a}^+_{\mathbf{k}}\hat{a}^-_{\mathbf{k}} + \delta^{(3)}(0)\right)E_k\right], \tag{6.34}$$

where

$$E_k(\eta) \equiv \left|v'_k\right|^2 + \omega_k^2(\eta)\left|v_k\right|^2, \tag{6.35}$$

$$F_k(\eta) \equiv v'^2_k + \omega_k^2(\eta)v_k^2. \tag{6.36}$$

Exercise 6.6
Derive (6.34).

Since $\hat{a}^-_{\mathbf{k}}\left|_{(v)}0\right\rangle = 0$, the expectation value of $\hat{H}(\eta_0)$ in the state $\left|_{(v)}0\right\rangle$ is

$$\left\langle_{(v)}0\right|\hat{H}(\eta_0)\left|_{(v)}0\right\rangle = \frac{1}{4}\delta^{(3)}(0)\int d^3\mathbf{k}E_k(\eta_0).$$

As discussed above, the divergent factor $\delta^{(3)}(0)$ is a harmless manifestation of the infinite total volume of space. The energy density is then

$$\varepsilon(\eta_0) = \frac{1}{4}\int d^3\mathbf{k}E_k(\eta_0) = \frac{1}{4}\int d^3\mathbf{k}\left(\left|v'_k(\eta_0)\right|^2 + \omega_k^2(\eta_0)\left|v_k(\eta_0)\right|^2\right), \tag{6.37}$$

and our task is to determine which mode functions $v_k(\eta)$ minimize $\varepsilon(\eta_0)$. It is clear that for each mode $\mathbf{k}$ its contribution to the energy must be minimized separately. Thus, for a given k we have to determine $v_k(\eta_0)$ and $v'_k(\eta_0)$ which minimize the expression

$$E_k(\eta_0) = \left|v'_k(\eta_0)\right|^2 + \omega_k^2(\eta_0)\left|v_k(\eta_0)\right|^2 \tag{6.38}$$

while obeying the normalization condition (6.22),

$$v'_k(\eta_0)v^*_k(\eta_0) - v_k(\eta_0)v^{*\prime}_k(\eta_0) = 2i. \tag{6.39}$$

Substituting

$$v_k = r_k \exp(i\alpha_k)$$

into (6.39), we infer that the real functions r_k and α_k obey

$$r_k^2\alpha'_k = 1. \tag{6.40}$$

With this relation we find that

$$E_k(\eta_0) = \left|v'_k\right|^2 + \omega_k^2\left|v_k\right|^2$$
$$= r'^2_k + r_k^2\alpha'^2_k + \omega_k^2 r_k^2 = r'^2_k + \frac{1}{r_k^2} + \omega_k^2 r_k^2 \tag{6.41}$$

is minimized when $r_k'(\eta_0) = 0$ and $r_k(\eta_0) = \omega_k^{-1/2}(\eta_0)$. We thus find that the initial conditions

$$v_k(\eta_0) = \frac{1}{\sqrt{\omega_k(\eta_0)}} e^{i\alpha_k(\eta_0)}, \quad v_k'(\eta_0) = i\sqrt{\omega_k(\eta_0)} e^{i\alpha_k(\eta_0)} = i\omega_k v_k(\eta_0), \quad (6.42)$$

select the preferred mode functions which determine the vacuum (the lowest-energy state) at a particular moment of time η_0. Although the phase factors $\alpha_k(\eta_0)$ remain undetermined, they are irrelevant and we can set them to zero. Note that the above considerations are valid only if $\omega_k^2 > 0$. For $\omega_k^2(\eta_0) < 0$ the function E_k has no minimum. In this case the instantaneous lowest-energy vacuum does not exist.

Remark: Hamiltonian diagonalization The mode functions satisfying the conditions (6.42) define a certain set of operators $\hat{a}_{\mathbf{k}}^{\pm}$ and the corresponding vacuum $\left|_{\eta_0} 0\right\rangle$. For these mode functions one finds $E_k(\eta_0) = 2\omega_k(\eta_0)$ and $F_k(\eta_0) = 0$, so the Hamiltonian at time η_0 is

$$\hat{H}(\eta_0) = \int d^3\mathbf{k}\, \omega_k(\eta_0) \left[\hat{a}_{\mathbf{k}}^{+} \hat{a}_{\mathbf{k}}^{-} + \frac{1}{2}\delta^{(3)}(0) \right]. \quad (6.43)$$

At $\eta = \eta_0$ this Hamiltonian is diagonal in the eigenbasis of the occupation number operators $\hat{N}_{\mathbf{k}} = \hat{a}_{\mathbf{k}}^{+}\hat{a}_{\mathbf{k}}^{-}$, which consists of the vacuum state $\left|_{\eta_0} 0\right\rangle$ and the corresponding excited states. Accordingly, the state $\left|_{\eta_0} 0\right\rangle$ is sometimes called the *vacuum of instantaneous Hamiltonian diagonalization.* The vacuum states at two different moments of time are related by Bogolyubov coefficients α_k and β_k, so particles are produced if $\beta_k \neq 0$.

Remark: zero-point energy As before, the zero-point energy density of the quantum field in the vacuum state $\left|_{\eta_0} 0\right\rangle$ is divergent,

$$\frac{1}{4}\int d^3\mathbf{k}\, E_k(\eta_0) = \frac{1}{2}\int d^3\mathbf{k}\, \omega_k(\eta_0).$$

This quantity is time-dependent and cannot be simply subtracted. A more sophisticated renormalization procedure (developed in Part II of this book) is needed to obtain the correct value of the energy density.

For a scalar field in Minkowski spacetime, ω_k is time-independent and the prescription (6.42) yields the standard mode functions (4.31), which determine the time-independent vacuum state. But if ω_k changes with time then the mode functions satisfying (6.42) at $\eta = \eta_0$ will generally differ from the mode functions that satisfy the same conditions at a different time $\eta_1 \neq \eta_0$. In other words, the vacua $\left|_{\eta_0} 0\right\rangle$ and $\left|_{\eta_1} 0\right\rangle$ are different and the state $\left|_{\eta_0} 0\right\rangle$ is not the lowest-energy state at a later moment of time η_1. In this case there are no states which remain eigenstates of the Hamiltonian at all times. This can be easily seen from (6.34).

A vacuum state could remain an eigenstate of the Hamiltonian only if $F_k = 0$ for all η, i.e.

$$F_k(\eta) = \left(v_k'\right)^2 + \omega_k^2(\eta) v_k^2 = 0.$$

This differential equation has the exact solutions,

$$v_k(\eta) = C \exp\left[\pm i \int \omega_k(\eta) d\eta\right],$$

which do not satisfy the mode function equation (6.21) if $\omega_k(\eta)$ depends on time.

The operators $\hat{a}_{\mathbf{k}}^{\pm}(\eta_0)$ and $\hat{a}_{\mathbf{k}}^{\pm}(\eta_1)$ defining the instantaneous vacuum states $\left|{}_{\eta_0}0\right\rangle$ and $\left|{}_{\eta_1}0\right\rangle$ at two different moments of time are related by a Bogolyubov transformation. The expectation value of the Hamiltonian $\hat{H}(\eta_1)$ in the vacuum state $\left|{}_{\eta_0}0\right\rangle$ is

$$\begin{aligned}\left\langle{}_{\eta_0}0\right| \hat{H}(\eta_1) \left|{}_{\eta_0}0\right\rangle &= \left\langle{}_{\eta_0}0\right| \int d^3\mathbf{k}\, \omega_k(\eta_1) \left[\hat{a}_{\mathbf{k}}^{+}(\eta_1)\hat{a}_{\mathbf{k}}^{-}(\eta_1) + \tfrac{1}{2}\delta^{(3)}(0)\right] \left|{}_{\eta_0}0\right\rangle \\ &= \delta^3(0) \int d^3\mathbf{k}\, \omega_k(\eta_1) \left[\tfrac{1}{2} + |\beta_k|^2\right],\end{aligned}$$

where β_k is the corresponding Bogolyubov coefficient. Unless $\beta_k = 0$ for all k, this energy is larger than the minimum possible value and hence the state $\left|{}_{\eta_0}0\right\rangle$ contains particles at time η_1.

Remark: minimal fluctuations The value of the mode function $v_k(\eta_0)$ can be chosen to be arbitrarily small without violating the normalization condition (6.22). However, in this case $v_k'(\eta_0)$ must be very large. This is the consequence of the Heisenberg uncertainty relation. In this case v_k would acquire large values within a short time, leading to large field fluctuations. In the lowest-energy state both $v_k(\eta_0)$ and $v_k'(\eta_0)$ are optimized so that the generation of large fluctuations within a short time is avoided. In this sense the vacuum state is the state with the minimal quantum fluctuations.

6.5.2 Ambiguity of the vacuum state

Minimization of the instantaneous energy is not the only possible way to define the "vacuum state" and there is no unique "best" physical prescription for choosing the vacuum state of a field in a general curved spacetime. The physical reason for this ambiguity is easy to understand. The usual definitions of the vacuum and particle states in Minkowski spacetime are based on a decomposition of fields into plane waves $\exp(i\mathbf{kx} - i\omega_k t)$. A localized particle with momentum k is described by a wavepacket with a momentum spread Δk and the particle momentum is well-defined only if $\Delta k \ll k$. The spatial size λ of the wavepacket is inversely proportional to Δk, so that $\lambda \sim 1/\Delta k$, and therefore $\lambda \gg 1/k$. However, when the geometry of a curved spacetime varies significantly across a region of size λ, the plane waves are not a good approximation to the solution of the wave equation and the usual definition of a particle with momentum k fails. This definition is meaningful only if the curvature scale (the distance below which the spacetime can be well approximated by Minkowski space) exceeds k^{-1}. Note that the relevant quantity is the four-dimensional curvature. Therefore, even in a spatially flat Friedmann universe the vacuum and particle states are not always well-defined for some modes. For example, those modes of the scalar field for which the squared frequency,

$$\omega_k^2(\eta) = k^2 + m^2a^2 - \frac{a''}{a},$$

is negative, do not oscillate and for them the analogy with a harmonic oscillator breaks down. Formally, even for $\omega_k^2 < 0$ mode expansions still make sense, but the interpretation of the corresponding states in Hilbert space in terms of physical particles is problematic. In particular, the "excited" states can have negative energy. Moreover, an eigenstate with the lowest instantaneous energy does not exist for such modes: For $\omega_k^2 < 0$ the condition $F_k(\eta_0) = 0$ contradicts the normalization condition (6.22) and hence the lowest energy eigenstate cannot be defined.

Further complications arise in curved spacetimes without symmetries. We shall see in Chapter 8 that an accelerated detector in a flat spacetime registers particles even when the field is in the true Minkowski vacuum state. Thus the definition of particles depends in general on the coordinate system which is preferred "from the point of view" of a detector. In a curved spacetime there is no a priori preferable coordinate system. Moreover, in the presence of gravity the energy is not necessarily bounded from below and the definition of the "true" vacuum state as the lowest energy state can therefore also fail.

Remark: short distances We have seen that the minimal energy state does not exist for modes with $\omega_k^2 < 0$. However, because $\omega_k^2 = k^2 + m_{\text{eff}}^2(\eta)$, modes with large enough k, namely,

$$k^2 > k_{\text{min}}^2 \equiv -m_{\text{eff}}^2 = \frac{a''}{a} - m^2a^2, \tag{6.44}$$

have positive $\omega_k^2 > 0$ even if $m_{\text{eff}}^2 < 0$, and therefore the lowest-energy state is well-defined for these modes. In cosmological applications, a negative m_{eff}^2 can arise because of the field interaction with the gravitational background. In such cases a natural length scale is the radius of curvature and ω_k^2 is negative only for modes exceeding the curvature scale. On much shorter scales, the spacetime can be treated as approximately flat. Therefore the field modes with wavelengths much smaller than the curvature radius are almost unaffected by gravitation. On very small scales, corresponding to large k, we can neglect $|m_{\rm eff}| \ll k$ and set $\omega_k \approx k$. Then the mode functions are those given in (4.31),

$$v_k(\eta) \approx \frac{1}{\sqrt{k}} e^{ik\eta}. \tag{6.45}$$

This leads to a natural definition of the minimal excitation state, which is unambiguous in the leading order and adequate only on small scales, $L \ll L_{\max} \sim k_{\min}^{-1} \sim |m_{\rm eff}|^{-1}$.

Remark: adiabatic vacuum As we have noted above, the notion of the particle in an arbitrary curved spacetime does not have an absolute meaning. Instead one has to consider the detector response which can be unambiguously determined for a given quantum state of the fields. Nevertheless sometimes it is useful to have "an approximate particle definition" which suits our intuition in the best possible way. In spacetimes with slowly changing geometry, the so-called adiabatic vacuum leads sometimes to a more meaningful notion of particles compared to the instantaneous vacuum prescription. In particular, in anisotropic universes a procedure based on the adiabatic vacuum allows one to separate the non-local contribution to the energy-momentum tensor resulting from the "particle production" from the local vacuum polarization effects in a more meaningful manner.

The adiabatic vacuum prescription relies on the WKB approximation for the solution of equation (6.21) in the case of slowly varying $\omega_k^2(\eta)$. Substituting the ansatz

$$v_k(\eta) = \frac{1}{\sqrt{W_k(\eta)}} \exp\left[i \int_{\eta_0}^{\eta} W_k(\eta) d\eta\right] \tag{6.46}$$

into (6.21) we find that the function $W_k(\eta)$ must obey the nonlinear equation

$$W_k^2 = \omega_k^2 - \frac{1}{2}\left[\frac{W_k''}{W_k} - \frac{3}{2}\left(\frac{W_k'}{W_k}\right)^2\right]. \tag{6.47}$$

Let us consider the case when ω_k is a slowly varying function of time. More precisely, we assume that ω_k and all its derivatives change substantially, i.e. $\Delta\omega_k/\omega_k \sim O(1)$, only during time intervals $T \gg 1/\omega_k$. In this case, equation (6.47) can be used as a recurrence relation which allows us to find a particular solution for W_k in the form of the asymptotic series in the powers of small parameter $(\omega_k T)^{-1}$. For example, to zeroth order in $(\omega_k T)^{-1}$ we have

$${}^{(0)}W_k = \omega_k,$$

while to second order

$${}^{(2)}W_k = \omega_k \left(1 - \frac{1}{4}\frac{\omega_k''}{\omega_k^3} + \frac{3}{8}\frac{\omega_k'^2}{\omega_k^4}\right).$$

In principle one could find ${}^{(N)}W_k$ to an arbitrary order N. However the series obtained is asymptotic, and so the accuracy of the approximation reaches an optimum value at a particular N and subsequently becomes worse as N grows. Substituting ${}^{(N)}W_k$ in (6.46) we obtain an approximate WKB solution $v_k^{(N)}(\eta)$ of the mode equation (6.21) to adiabatic order N. Then the mode functions $v_k(\eta)$ determining the *adiabatic vacuum of order N at a particular time* η_0 are defined by the requirement that the *exact* solution $v_k(\eta)$ of equation (6.21) satisfies the following initial conditions,

$$v_k(\eta_0) = v_k^{(N)}(\eta_0), \quad v_k'(\eta_0) = v_k^{(N)\prime}(\eta_0).$$

All vacuum prescriptions agree if $\omega_k(\eta)$ is exactly constant. In particular, in the case when $\omega_k(\eta)$ tends to a constant both in the remote past $(\eta \ll \eta_1)$ and in the future $(\eta \gg \eta_2)$ one can unambiguously define "in" and "out" particle states in the past and future respectively. If the frequency $\omega_k(\eta)$ is time-dependent within some interval $\eta_1 < \eta < \eta_2$ then the positive-frequency solution for $\eta \ll \eta_1$ evolves to a mixture of positive and negative frequency solutions for $\eta \gg \eta_2$. As a consequence, particles are produced and the number density of these particles can be unambiguously determined in the "out" region $\eta \gg \eta_2$. On the other hand, the notion of the particle is ambiguous in the intermediate regime, $\eta_1 < \eta < \eta_2$, when ω_k is time-dependent. The reason is that in this case the vacuum fluctuations are not only "excited" but also "deformed" by the external field. This latter effect is called the *vacuum polarization.* There is no unique way to separate the "particles" and the vacuum polarization contributions in the total energy-momentum tensor. However, this does not lead to ambiguities in physical predictions because only the total energy-momentum tensor is relevant as the source of the gravitational field. The response of a specific particle detector can also be unambiguously determined given a quantum state of the field.

Thus, the absence of a generally valid definition of the vacuum and particle states does not impair our ability to make predictions for specific observable quantities in a curved spacetime. All well-posed physical questions can always be unambiguously answered even in the absence of such a definition.

Remark: a quantum-mechanical analogy Note that the stationary Schrödinger equation for a particle in a one-dimensional potential $V(x)$,

$$\frac{d^2\psi}{dx^2} + (E - V(x))\,\psi = 0,$$

coincides with the mode equation (6.21) after we replace the spatial coordinate x by the time η and substitute $\omega_k^2(\eta)$ for $E - V(x)$. The wave function ψ then "plays the role" of the mode function v_k. This allows us to draw a formal mathematical analogy between the problem of particle creation and the problem of a quantum-mechanical penetration through a potential barrier. Considering a plane wave with an amplitude α falling onto the potential barrier from the right (see Fig. 6.1) we find that the incident wave "splits" into reflected and transmitted waves. Normalizing the amplitude of the transmitted wave to unity $(T = 1)$ we obtain from the conservation of probability that $|\alpha|^2 = |\beta|^2 + 1$, the condition analogous to (6.25). The transmitted wave in this consideration corresponds to the initial vacuum fluctuations in the problem of particle creation, while the reflected

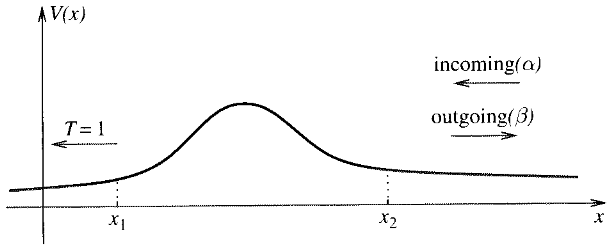

Fig. 6.1 Quantum-mechanical analogy: motion in a potential $V(x)$.

wave "describes the produced particles." We would like to stress once more that this analogy is entirely formal and is useful only to those who have a solid intuition for the corresponding quantum-mechanical problem.

6.6 Amplitude of quantum fluctuations

Correlation function Given a quantum state of the field $|\psi\rangle$, the amplitude of quantum fluctuations is always well-defined irrespective of whether the particle interpretation of the field is available. Let us consider the equal-time correlation function $\langle\psi|\,\hat{\chi}(\mathbf{x},\eta)\,\hat{\chi}(\mathbf{y},\eta)\,|\psi\rangle$. For a "vacuum state" $|0\rangle$ determined by a set of mode functions $v_k(\eta)$, we obtain

$$\langle 0|\,\hat{\chi}(\mathbf{x},\eta)\,\hat{\chi}(\mathbf{y},\eta)\,|0\rangle = \frac{1}{4\pi^2}\int_0^\infty k^2 dk\,|v_k(\eta)|^2\,\frac{\sin kL}{kL}, \tag{6.48}$$

where $L \equiv |\mathbf{x}-\mathbf{y}|$.

Exercise 6.7
Derive (6.48) using the mode expansion (6.20).

Generically the main contribution to the integral in (6.48) comes from wavenumbers $k \sim L^{-1}$, and therefore the magnitude of the correlation function can be estimated as

$$\langle 0|\,\hat{\chi}(\mathbf{x},\eta)\,\hat{\chi}(\mathbf{y},\eta)\,|0\rangle \sim k^3\,|v_k|^2, \tag{6.49}$$

with $k \sim L^{-1}$. Note that in the Friedmann universe the comoving coordinate distance $L = |\mathbf{x}-\mathbf{y}|$ is related to the physical distance L_p as $L_p = a(\eta)L$, where $a(\eta)$ is the scale factor. The field χ is related to the original, physical field ϕ by $\phi = \chi/a(\eta)$.

Fluctuations of spatially averaged fields One can consider a field operator averaged over a region of size L (e.g. a cube with sides $L \times L \times L$),

$$\hat{\chi}_L(\eta) \equiv \frac{1}{L^3}\int_{L\times L\times L} \hat{\chi}(\mathbf{x},\eta)\,d^3\mathbf{x},$$

and calculate

$$\delta\chi_L^2(\eta) \equiv \langle\psi| [\hat{\chi}_L(\eta)]^2 |\psi\rangle .$$

This is another way to characterize the typical fluctuations on scales L. Sometimes instead of integrating over the box with the sharp boundaries it is more convenient to define the *window-averaged operator*

$$\hat{\chi}_L(\eta) \equiv \int \hat{\chi}(\mathbf{x}, \eta)\, W_L(\mathbf{x}) d^3\mathbf{x},$$

where $W_L(\mathbf{x})$ is a *window function* $W(\mathbf{x})$ which is of order 1 for $|\mathbf{x}| \lesssim L$ and rapidly decays for $|\mathbf{x}| \gg L$. This function must satisfy the normalization condition

$$\int W(\mathbf{x})\, d^3\mathbf{x} = 1. \tag{6.50}$$

The prototypical example of a window function is the *Gaussian* function

$$W_L(\mathbf{x}) = \frac{1}{(2\pi)^{3/2}L^3} \exp\left(-\frac{|\mathbf{x}|^2}{2L^2}\right)$$

which selects $|\mathbf{x}| \lesssim L$. In the general case it is rather natural to select a window function with the following scaling properties:

$$W_{L'}(\mathbf{x}) \equiv \frac{L^3}{L'^3} W_L\left(\frac{L}{L'}\mathbf{x}\right).$$

In this case

$$\int W_L(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} d^3\mathbf{x} = w(\mathbf{k}L),$$

and the Fourier image $w(\mathbf{k}L)$ satisfies $w|_{\mathbf{k}=0} = 1$ and decays rapidly for $|\mathbf{k}| \gtrsim L^{-1}$.

Given the mode expansion (6.20) for the field operator $\hat{\chi}(\mathbf{x}, \eta)$, after straightforward algebra we find

$$\delta\chi_L^2(\eta) = \langle 0| \left[\int d^3\mathbf{x}\, W_L(\mathbf{x})\hat{\chi}(\mathbf{x}, \eta)\right]^2 |0\rangle = \frac{1}{2}\int |v_k|^2 |w(\mathbf{k}L)|^2 \frac{d^3\mathbf{k}}{(2\pi)^3}.$$

Since the function $w(\mathbf{k}L)$ is of order unity for $|\mathbf{k}| \lesssim L^{-1}$ and quickly decays for $|\mathbf{k}| \gtrsim L^{-1}$, one can estimate the above integral as

$$\int |v_k|^2 |w(\mathbf{k}L)|^2 \frac{d^3\mathbf{k}}{(2\pi)^3} \sim \int_0^{L^{-1}} k^2 |v_k|^2 dk \sim \frac{1}{L^3} |v_k|^2, \quad k \sim L^{-1}.$$

Thus the amplitude of fluctuations $\delta\chi_L$ is of order

$$\delta\chi_L^2 \sim k^3 |v_k|^2, \tag{6.51}$$

where $k \sim L^{-1}$.

Comparing (6.49) and (6.51), we see that the correlation function and the mean square fluctuation both have the same order of magnitude and both characterize the typical amplitude of quantum fluctuations on scales L. Therefore we refer to

$$\delta(k) \equiv \frac{1}{2\pi} k^{3/2} |v_k| \tag{6.52}$$

as the spectrum of quantum fluctuations.

6.6.1 Comparing fluctuations in the vacuum and excited states

Intuitively one may expect that the fluctuations in an excited state are larger than those in the vacuum state. To demonstrate this, let us compare the fluctuations of a scalar field for the vacuum and excited states in Minkowski spacetime. The spectrum of the vacuum fluctuations in Minkowski spacetime was already calculated in Chapter 4, equation (4.34), and the result is

$$\delta_{\text{vac}}(k) = \frac{1}{2\pi} \frac{k^{3/2}}{\sqrt{\omega_k}} = \frac{1}{2\pi} \frac{k^{3/2}}{(k^2 + m^2)^{1/4}}. \tag{6.53}$$

This time-independent spectrum is sketched in Fig. 6.2. When measured with a high-resolution device (small L or large k), the field shows large fluctuations. On the other hand, if the field is averaged over a large volume ($L \to \infty$), the amplitude of fluctuations tends to zero.

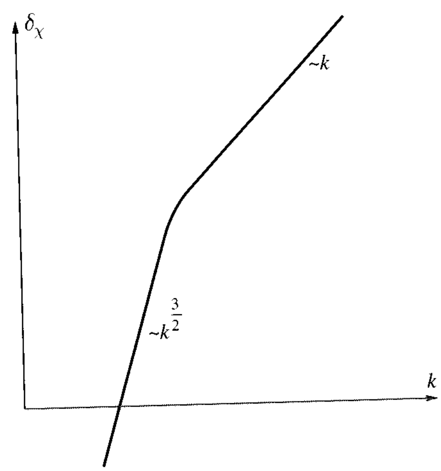

Fig. 6.2 A sketch of the spectrum of fluctuations $\delta\chi_L$ in Minkowski space; $L \equiv 2\pi k^{-1}$. (The logarithmic scaling is used for both axes.)

Let us consider the (nonvacuum) state $|b\rangle$ annihilated by all operators $\hat{b}_{\mathbf{k}}^-$ defined via the field operator expansion

$$\hat{\chi} = \frac{1}{\sqrt{2}} \int \left(e^{i\mathbf{k}\cdot\mathbf{x}} v_k^* \hat{b}_{\mathbf{k}}^- + e^{-i\mathbf{k}\cdot\mathbf{x}} v_k \hat{b}_{\mathbf{k}}^+ \right) \frac{d^3\mathbf{k}}{(2\pi)^{3/2}},$$

where

$$v_k = \frac{1}{\sqrt{\omega_k}} \left(\alpha_k e^{i\omega_k \eta} + \beta_k e^{-i\omega_k \eta} \right). \tag{6.54}$$

Substituting (6.54) in (6.52), we find the spectrum of fluctuations in the state $|b\rangle$:

$$\delta_b(k) = \frac{1}{2\pi} \frac{k^{3/2}}{\sqrt{\omega_k}} \left[|\alpha_k|^2 + |\beta_k|^2 - 2\mathrm{Re}\left(\alpha_k \beta_k e^{2i\omega_k \eta} \right) \right]^{1/2}. \tag{6.55}$$

Thus for the ratio of the amplitudes we obtain

$$\frac{\delta_b^2}{\delta_{\mathrm{vac}}^2} = 1 + 2|\beta_k|^2 - 2\mathrm{Re}\left(\alpha_k \beta_k e^{2i\omega_k \eta} \right). \tag{6.56}$$

After averaging over a sufficiently long time interval, $\Delta\eta \gg \omega_k^{-1}$, the oscillating term $\mathrm{Re}\left(\alpha_k \beta_k e^{2i\omega_k \eta}\right)$ vanishes and the result in (6.56) simply reduces to $1 + 2|\beta_k|^2$.

6.7 An example of particle production

For illustrative purposes we now perform explicit calculations for a rather artificial but simple case when the effective mass of the scalar field changes as follows,

$$m_{\mathrm{eff}}^2(\eta) = \begin{cases} m_0^2, & \eta < 0 \text{ and } \eta > \eta_1; \\ -m_0^2, & 0 < \eta < \eta_1. \end{cases} \tag{6.57}$$

In the regions $\eta < 0$ and $\eta > \eta_1$ the vacuum states are well-defined; they are called the "in" vacuum $|0_{\mathrm{in}}\rangle$ and the "out" vacuum $|0_{\mathrm{out}}\rangle$ respectively. We assume that initially (for $\eta < 0$) the scalar field is the "in" vacuum state and compute (a) the mean particle number, (b) the mean energy of produced particles, and (c) the spectrum of quantum fluctuations for $\eta > \eta_1$.

Mode functions The "in" and "out" vacuum states are entirely determined by specifying the negative frequency mode functions, which are

$$v_k^{(\mathrm{in})}(\eta) = \frac{1}{\sqrt{\omega_k}} e^{i\omega_k \eta}, \tag{6.58}$$

for $\eta < 0$ and

$$v_k^{(\mathrm{out})}(\eta) = \frac{1}{\sqrt{\omega_k}} e^{i(\eta - \eta_1)\omega_k}, \tag{6.59}$$

for $\eta > \eta_1$, where $\omega_k \equiv \sqrt{k^2 + m_0^2}$.

Since $\omega_k^2(\eta) = k^2 + m_{\text{eff}}^2(\eta)$ changes at $\eta = 0$ and $\eta = \eta_1$, the mode functions $v_k^{(\text{in})}(\eta)$ evolve into superposition of the negative and positive frequencies for $\eta > \eta_1$:

$$v_k^{(\text{in})}(\eta) = \frac{1}{\sqrt{\omega_k}}\left[\alpha_k^* e^{i\omega_k(\eta-\eta_1)} + \beta_k^* e^{-i\omega_k(\eta-\eta_1)}\right]. \tag{6.60}$$

The Bogolyubov coefficients α_k, β_k are determined by the requirement that the solution and its first derivative must be continuous at $\eta = 0$ and $\eta = \eta_1$. The result is

$$\alpha_k = \frac{e^{-i\Omega_k\eta_1}}{4}\left(\sqrt{\frac{\omega_k}{\Omega_k}} + \sqrt{\frac{\Omega_k}{\omega_k}}\right)^2 - \frac{e^{i\Omega_k\eta_1}}{4}\left(\sqrt{\frac{\omega_k}{\Omega_k}} - \sqrt{\frac{\Omega_k}{\omega_k}}\right)^2,$$

$$\beta_k = \frac{1}{4}\left(\frac{\Omega_k}{\omega_k} - \frac{\omega_k}{\Omega_k}\right)\left(e^{i\Omega_k\eta_1} - e^{-i\Omega_k\eta_1}\right) = \frac{i}{2}\left(\frac{\Omega_k}{\omega_k} - \frac{\omega_k}{\Omega_k}\right)\sin(\Omega_k\eta_1),$$

where $\omega_k \equiv \sqrt{k^2 + m_0^2}$ and $\Omega_k \equiv \sqrt{k^2 - m_0^2}$.

Exercise 6.8

Derive the above expressions for the Bogolyubov coefficients.

Particle number density For $\eta > \eta_1$ the state $|0_{\text{in}}\rangle$ is different from the true vacuum state $|0_{\text{out}}\rangle$ and so the state $|0_{\text{in}}\rangle$ contains particles. The mean particle number density in a mode $\mathbf{k}$ is

$$n_k = |\beta_k|^2 = \frac{m_0^4}{|k^4 - m_0^4|}\left|\sin\left(\eta_1\sqrt{k^2 - m_0^2}\right)\right|^2. \tag{6.61}$$

Note that this expression remains finite as $k \to m_0$. Let us consider separately two limiting cases: $k \gg m_0$ (ultrarelativistic particles) and $k \ll m_0$ (nonrelativistic particles).

For $k \gg m_0$, we have $\omega_k \approx \Omega_k$ and assuming that $m_0\eta_1$ is not too large one can expand (6.61) in powers of the small parameter (m_0/k). After some algebra we obtain

$$n_k = \frac{m_0^4}{k^4}\sin^2(k\eta_1) + O\left(\frac{m_0^5}{k^5}\right). \tag{6.62}$$

It follows that $n_k \ll 1$ or, in other words, very few relativistic particles are created.

The situation is different for $k \ll m_0$. In this case $\sqrt{k^2 - m_0^2} \approx im_0$ is imaginary and we obtain

$$n_k \sim \sinh^2(m_0\eta_1). \tag{6.63}$$

If $m_0\eta_1 \gg 1$, the density of the produced particles is exponentially large.

Particle energy density Since $n_k \sim k^{-4}$ for $k \to \infty$, the energy density of the produced particles,

$$\varepsilon_0 = \int n_k \omega_k d^3\mathbf{k}, \tag{6.64}$$

logarithmically diverges for large k (ultraviolet limit). This divergence is, however, entirely due to the discontinuous change of $m_{\mathrm{eff}}^2(\eta)$ and it disappears for any smooth function $m_{\mathrm{eff}}^2(\eta)$. Therefore, we ignore this divergence and assume that there is an ultraviolet cutoff at some $k_{\max}$. Then for $m_0\eta_1 \gg 1$, the main contribution to the integral comes from the modes with $k \lesssim m_0$ for which $|\omega_k| \sim m_0$. Using (6.63) we can estimate the energy density of the produced particles as

$$\varepsilon_0 \sim m_0 \int_0^{m_0} dk\, k^2 \exp(2m_0\eta_1) \sim m_0^4 \exp(2m_0\eta_1).$$

Exercise 6.9*

Assuming that the integral in (6.64) is performed over the range $0 < k < k_{\max}$, show that for $m_0\eta_1 \gg 1$ the dominant contribution to the integral comes from $k \approx \sqrt{m_0/\eta_1}$ and derive a more precise estimate for ε_0:

$$\varepsilon_0 \propto \frac{m_0^4}{(m_0\eta_1)^{3/2}} \exp(2m_0\eta_1).$$

Amplitude of fluctuations Neglecting the oscillating term in (6.55) we immediately obtain the following estimate for the amplitude of quantum fluctuations at $\eta > \eta_1$:

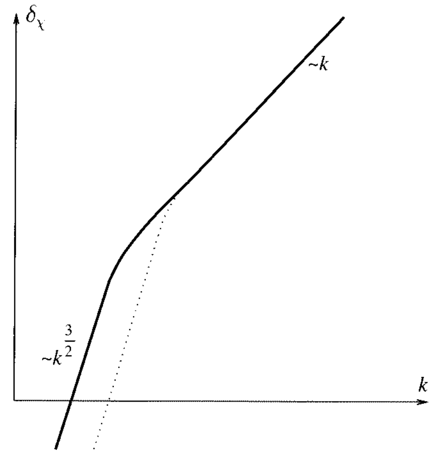

Fig. 6.3 A sketch of the spectrum $\delta\chi_L$ after particle creation; $L \equiv 2\pi k^{-1}$. (The logarithmic scaling is used for both axes.) The dotted line is the spectrum in Minkowski space.

$$\delta \sim \frac{k^{3/2}}{\sqrt{\omega_k}}\left(1+2\left|\beta_k\right|^2\right)^{1/2} \sim \begin{cases} k, & k \gg m_0; \\ k^{3/2} m_0^{-1/2} \exp\left(m_0 \eta_1\right), & k \ll m_0. \end{cases}$$

Thus, we see that on large scales the amplitude of fluctuations is enhanced by the factor $\exp(m_0\eta_1)$ compared to the amplitude of the vacuum fluctuations (see Fig. 6.3).