5

Reminder: classical fields

Summary Action functional in general. Minimal and conformal coupling of scalar field to gravity. Internal symmetries and gauge fields. Gravitational field. The energy-momentum tensor. Conservation laws.

5.1 The action functional

In this book we mainly consider a quantum scalar field interacting with the classical gravitational or electromagnetic fields. To determine the admissible form of their couplings we have to recall the basic principles of classical field theory.

A theory of a classical field $\phi_i(x)$ is based on the action

$$S[\phi] = \int d^4x\, \mathcal{L}(\phi_i, \partial_\mu \phi_i, \ldots) \tag{5.1}$$

where i is an "internal index," $\partial_\mu \phi_i \equiv \partial \phi_i / \partial x^\mu$, and the Lagrangian density $\mathcal{L}$ depends on the field strength and its derivatives. The variable ϕ_i can designate a real or complex scalar field, a vector field, or a tensor field. For instance, in the case of the gravitational field $\phi_i \equiv g_{\alpha\beta}(x^\gamma)$, where $g_{\alpha\beta}$ is the metric. Consideration of fermionic fields is beyond the scope of this book.

Choosing the action functional The action functional is usually chosen in accordance with the following guiding principles:

(i) The action must be real-valued because otherwise the total probability is not conserved in the corresponding quantum theory.

(ii) Local theories have so far been successful in describing experiments and therefore the action is usually taken as a *local functional* of the fields and their derivatives. The Lagrangian density $\mathcal{L}$ is then a *function* of the fields and their derivatives. Otherwise, as for example in the case

$$\int d^4x\, d^4x'\, \phi^{,\mu}(x - x')\phi_{,\mu}(x'),$$

the Lagrangian density is also a functional and the values of the field at separated points x and x' are "coupled."

(iii) Usually it is sufficient to specify the initial conditions for the field itself and at most for its first derivatives in order to unambiguously predict the subsequent evolution of the field. This means that the equations of motion contain derivatives of at most second order and hence we can restrict ourselves to actions which depend only on the field strength and its first derivatives, that is, $\mathcal{L} = \mathcal{L}(\phi_i, \phi_{i,\mu})$.

(iv) In the absence of gravity, the action must be Poincaré-invariant in order to respect the translational and Lorentz invariance of the flat spacetime. This requirement strongly constrains possible Lagrangians. In particular, the Lagrangian density $\mathcal{L}$ cannot depend explicitly on $\mathbf{x}$ or t.

(v) In an arbitrary curved spacetime, the action must be invariant with respect to general coordinate transformations since physical properties of the system are independent of the coordinate system used ("geometrical character" of nature).

(vi) The conservation of different charges is usually related to the existence of some internal symmetries, among which gauge symmetry is of particular importance. In such cases the action must respect these symmetries. For example, the Lagrangian describing an electrically charged complex scalar field must be invariant with respect to $U(1)$ local gauge transformations because otherwise the electric charge would not be conserved. The Standard Model of particle physics is based on $SU(3) \times SU(2) \times U(1)$ group of local gauge transformations.

As we will see shortly, the above requirements usually suffice to fix the coupling of the different fields almost unambiguously.

Equations of motion The requirement that the action takes an extremum value for a classically allowed field configuration $\phi_i(x)$ leads to the Euler–Lagrange equations of motion,

$$\frac{\delta S}{\delta \phi_i(x)} = 0.$$

For a Lagrangian density $\mathcal{L} = \mathcal{L}(\phi_i, \phi_{i,\mu})$, the variation of the action is

$$\delta S = \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi_i} - \frac{\partial}{\partial x^\mu} \frac{\partial \mathcal{L}}{\partial \phi_{i,\mu}} \right) \delta\phi_i(x) + O\left([\delta\phi]^2\right),$$

and so the equations of motion are

$$\frac{\delta S[\phi]}{\delta \phi_i(x)} = \frac{\partial \mathcal{L}}{\partial \phi_i} - \frac{\partial}{\partial x^\mu} \frac{\partial \mathcal{L}}{\partial \phi_{i,\mu}} = 0. \tag{5.2}$$

Here summation over μ is implied and we have assumed that the boundary terms vanish sufficiently rapidly as $|\mathbf{x}| \to \infty$, $|t| \to \infty$.

The formula (5.2) holds for all Lagrangians that depend on the field strength and at most on its first derivatives. If the Lagrangian contains second-order derivatives such as $\phi_{,\mu\nu}$, the corresponding equations of motion are generally of third or fourth order.

5.2 Real scalar field and its coupling to the gravity

The simplest relativistically invariant Lagrangian density for a real scalar field $\phi(x)$ in a flat spacetime takes the form

$$\mathcal{L}(\phi, \partial_\mu \phi) = \frac{1}{2}\eta^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - V(\phi), \tag{5.3}$$

where $\eta^{\mu\nu} \equiv \mathrm{diag}(1, -1, -1, -1)$ is the Minkowski metric and $V(\phi)$ is a potential that describes the self-interaction of the field. For a *free* (i.e. noninteracting) field of mass m the potential is

$$V(\phi) = \frac{1}{2}m^2\phi^2.$$

Note that a linear term $A\phi$ in the potential can be always removed by a field redefinition $\phi(x) = \tilde{\phi}(x) + \phi_0$.

To generalize the Lagrangian (5.3) to the case of a curved spacetime with an arbitrary metric $g_{\mu\nu}$, we have to:

(i) replace $\eta_{\mu\nu}$ with the metric $g_{\mu\nu}$;
(ii) replace ordinary derivatives by covariant derivatives (note that the first covariant derivative of a scalar function coincides with the ordinary derivative);
(iii) use the covariant volume element $d^4x\sqrt{-g}$, where $g \equiv \det g_{\mu\nu}$, instead of the usual volume element $d^3\mathbf{x}\, dt$.

The resulting action,

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - V(\phi)\right], \tag{5.4}$$

explicitly depends on $g_{\mu\nu}$ and describes a scalar field *minimally coupled* to the gravity. This coupling necessarily follows from the requirement of general covariance.

Remark: covariant volume element To understand the appearance of the factor $\sqrt{|g|}$ in the volume element, let us consider a two-dimensional Euclidean plane covered by curvilinear coordinates $\tilde{x}$, $\tilde{y}$. In these coordinates, the metric $g_{ij}(\tilde{x})$, where $i, j = 1, 2$, is generally different from the Euclidean metric δ_{ij}. Infinitesimal coordinate increments $d\tilde{x}$, $d\tilde{y}$ define an area element corresponding to the infinitesimal parallelogram spanned by the vectors $\mathbf{l}_1 = (d\tilde{x}, 0)$ and $\mathbf{l}_2 = (0, d\tilde{y})$. The length of the vector $\mathbf{l}_1$ is given by

$$|\mathbf{l}_1| = \sqrt{g_{ij}l_1^i l_1^j} = \sqrt{g_{11}}d\tilde{x}.$$

Similarly, we find $|\mathbf{l}_2| = \sqrt{g_{22}}d\tilde{y}$. The scalar product of the vectors $\mathbf{l}_1$ and $\mathbf{l}_2$ is

$$\mathbf{l}_1 \cdot \mathbf{l}_2 = g_{ij} l_1^i l_2^j = g_{12} d\tilde{x} d\tilde{y}.$$

On the other hand, one can express the scalar product $\mathbf{l}_1 \cdot \mathbf{l}_2$ through the angle θ between the vectors (according to the cosine theorem),

$$\mathbf{l}_1 \cdot \mathbf{l}_2 = |\mathbf{l}_1|\,|\mathbf{l}_2| \cos\theta = \sqrt{g_{11}g_{22}}\, d\tilde{x} d\tilde{y} \cos\theta.$$

Comparing the two expressions for $\mathbf{l}_1 \cdot \mathbf{l}_2$, we find that

$$\cos\theta = \frac{g_{12}}{\sqrt{g_{11}g_{22}}},$$

and the infinitesimal area dA of the parallelogram is

$$dA = |\mathbf{l}_1|\,|\mathbf{l}_2| \sin\theta = \sqrt{g_{11}g_{22} - (g_{12})^2}\, d\tilde{x} d\tilde{y} = \sqrt{\det g_{ij}}\, d\tilde{x} d\tilde{y}.$$

In n dimensions this formula is generalized to $dV = d^n x \sqrt{|g(x)|}$. In a four-dimensional spacetime where the metric has the signature $(+, -, -, -)$, the determinant g is always negative and hence the volume element is $d^4 x \sqrt{-g}$.

Nonminimal and conformal couplings The action can in principle contain additional terms which directly couple the fields to the curvature tensor $R_{\mu\nu\rho\sigma}$. Such couplings to the gravity are called *nonminimal* and they violate the strong equivalence principle, which states that *all local* effects of gravity must disappear in the local inertial frame. However, curvature does not vanish in a local inertial frame and hence influences the behavior of fields in theories with nonminimal coupling. However, the only criterion for the legitimacy of a theory is agreement with experiment. Theories violating the strong equivalence principle are allowed as long as they agree with available experiments.

The simplest action for a nonminimally coupled scalar field is

$$S = \int d^4 x \sqrt{-g} \left[\frac{1}{2} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - V(\phi) - \frac{\xi}{2} R \phi^2 \right], \tag{5.5}$$

where R is the Ricci curvature scalar and ξ is a constant parameter. The additional term induces a "mass" correction which is proportional to the scalar curvature. With $V = 0$ and $\xi = 1/6$, this theory has an additional symmetry, namely, the action (5.5) is invariant under conformal transformations,

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = \Omega^2(x) g_{\mu\nu}, \tag{5.6}$$

where the conformal factor $\Omega^2(x)$ is an arbitrary function.[1]

[1] Verifying the conformal invariance of the above action takes a fair amount of algebra. We omit the details of this calculation which can be found in Chapter 6 of the book by S. Fulling, *Aspects of Quantum Field Theory in Curved Space-Time* (Cambridge, 1989).

As we will see later, the energy-momentum tensor of a conformally invariant field is traceless in the classical theory. Conformal invariance has also another important aspect: In *conformally flat spacetimes*, where the metric can be written down as $g_{\mu\nu} = \Omega^2(x)\eta_{\mu\nu}$, the influence of the gravitational field on the behavior of the conformally invariant quantum field is greatly simplified. An important example of a conformally flat spacetime is the Friedmann universe.

The equation of motion for the real scalar field ϕ coupled to the gravity follows from the action (5.5),

$$\partial_\alpha \frac{\partial \mathcal{L}}{\partial \phi_{,\alpha}} - \frac{\partial \mathcal{L}}{\partial \phi} = \left(\sqrt{-g} g^{\alpha\beta} \phi_{,\beta}\right)_{,\alpha} + \left(\frac{\partial V}{\partial \phi} + \xi R \phi\right)\sqrt{-g} = 0. \tag{5.7}$$

For the minimally coupled scalar field we have $\xi = 0$.

Equation (5.7) can also be rewritten in a manifestly covariant form

$$\phi^{;\alpha}_{;\alpha} + \frac{\partial V}{\partial \phi} + \xi R \phi = 0, \tag{5.8}$$

where ; α denotes the covariant derivative with respect to the coordinate x^α.

5.3 Gauge invariance and coupling to the electromagnetic field

A real scalar field is electrically neutral and does not couple to the electromagnetic field. The conservation of the electric charge is related to the existence of internal symmetry associated with $U(1)$ gauge transformations. These transformations are implemented as multiplication of the field by $\exp(i\alpha)$, where α is an arbitrary real constant. Therefore they cannot be realized for the real scalar field and the electrically charged scalar field is necessarily described by the complex variable φ, or equivalently, by two real scalar fields.

The action

$$S[\phi] = \int d^4x \sqrt{-g}\left[\frac{1}{2} g^{\alpha\beta} \varphi_{,\alpha} \varphi^*_{,\beta} - V(\varphi\varphi^*)\right] \tag{5.9}$$

is clearly invariant with respect to the *global gauge transformation*

$$\varphi(x) \to \tilde{\varphi}(x) = e^{i\alpha} \varphi(x), \tag{5.10}$$

where α is a real constant, i.e. α is the same number for all points in the spacetime. This explains why transformation (5.10) is called global.

The minimal coupling of the charged scalar field with the electromagnetic field is unambiguously determined by the requirement of the *local* gauge invariance. Generalizing (5.10) to a *local gauge transformation*,

$$\varphi(x) \to \tilde{\varphi}(x) = e^{i\alpha(x)} \varphi(x), \tag{5.11}$$

where $\alpha(x)$ an arbitrary function of the spacetime coordinates, we find that

$$\varphi_{,\mu}(x) \to \tilde{\varphi}_{,\mu}(x) = e^{i\alpha(x)} \left(\varphi_{,\mu} + i\alpha_{,\mu}\varphi\right).$$

Hence, action (5.9) is not invariant under local gauge transformations. To regain its local gauge invariance we are forced to introduce an additional vector field A_μ called the *gauge field*, which compensates the extra term in the derivative $\varphi_{,\mu}$. (In the present case the field A_μ is interpreted as the electromagnetic field.) Then, replacing the ordinary derivatives ∂_μ by the gauge covariant derivatives D_μ,

$$\varphi_{,\mu} \to D_\mu\varphi \equiv \varphi_{,\mu} + iA_\mu\varphi, \tag{5.12}$$

and postulating for the field A^μ the transformation law

$$A_\mu \to \tilde{A}_\mu \equiv A_\mu - \alpha_{,\mu}, \tag{5.13}$$

we find

$$\tilde{D}_\mu\tilde{\varphi} = \left(\partial_\mu + i\tilde{A}_\mu\right)\left(e^{i\alpha(x)}\varphi\right) = e^{i\alpha(x)} D_\mu\varphi.$$

It follows that the action

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}g^{\alpha\beta}\left(D_\alpha\varphi\right)\left(D_\beta\varphi\right)^* - V\left(\varphi\varphi^*\right)\right] \tag{5.14}$$

$$= \int d^4x\sqrt{-g}\left[\frac{1}{2}g^{\alpha\beta}\varphi_{,\alpha}\varphi^*_{,\beta} - V\left(\varphi\varphi^*\right)\right.$$
$$\left. + \frac{1}{2}g^{\alpha\beta}iA_\alpha\left(\varphi\varphi^*_{,\beta} - \varphi^*\varphi_{,\beta}\right) + \frac{1}{2}g^{\alpha\beta}A_\alpha A_\beta\varphi\varphi^*\right] \tag{5.15}$$

is invariant under local gauge transformations (5.11)–(5.13). This action describes the minimal coupling of the charged scalar field to the electromagnetic field and to gravity. In the following chapters we consider the behavior of a quantum scalar field in the presence of the classical gravitational and electromagnetic fields.

5.4 Action for the gravitational and gauge fields

In the second part of the book we will need the action for the classical gravitational and electromagnetic fields themselves. The simplest possible action for the gravitational field is the *Einstein–Hilbert action*:

$$S^{\text{grav}} = -\frac{1}{16\pi G}\int d^4x\sqrt{-g}(R + 2\Lambda), \tag{5.16}$$

where G is Newton's gravitational constant, R is the Ricci scalar curvature and Λ is a constant parameter (the cosmological constant). The Einstein equations are obtained by extremizing this action with respect to $g^{\alpha\beta}$.

Exercise 5.1
Derive the Einstein equations in the absence of matter and cosmological constant using the *Palatini method:* assume that the metric $g_{\mu\nu}$ and the Christoffel symbol $\Gamma^{\mu}_{\alpha\beta}$ are independent and vary the action with respect to both of them.

Hint: The expression for the Ricci tensor in terms of $g_{\mu\nu}$ and $\Gamma^{\mu}_{\alpha\beta}$ is

$$R = g^{\alpha\beta} R_{\alpha\beta} = g^{\alpha\beta} \left(\partial_\mu \Gamma^{\mu}_{\alpha\beta} - \partial_\beta \Gamma^{\mu}_{\alpha\mu} + \Gamma^{\mu}_{\alpha\beta} \Gamma^{\nu}_{\mu\nu} - \Gamma^{\nu}_{\alpha\mu} \Gamma^{\mu}_{\beta\nu} \right). \tag{5.17}$$

First find the variation of $R\sqrt{-g}$ with respect to Γ and assuming that $\Gamma^{\mu}_{\alpha\beta}$ is symmetric in α,β establish the standard relation between Γ and g,

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} \left(g_{\alpha\nu,\beta} + g_{\beta\nu,\alpha} - g_{\alpha\beta,\nu} \right). \tag{5.18}$$

Then compute the variation of $R\sqrt{-g}$ with respect to $g^{\alpha\beta}$ and obtain the vacuum Einstein equation

$$\frac{\delta S^{\mathrm{grav}}}{\delta g^{\alpha\beta}} = -\frac{\sqrt{-g}}{16\pi G} \left(R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R \right) = 0. \tag{5.19}$$

Remark: higher-derivative gravity The Einstein equations are the only possible second-order covariant equations for the gravitational field. Any modification of the Einstein–Hilbert action in four dimensions leads to a higher-derivative gravity. At present, the Einstein theory is in a very good agreement with experiments. However, it is likely that this theory breaks down in regions with an extremely strong gravitational field, where the curvature radius is comparable to the Planck length. In this case, the correct gravitational action must contain additional terms quadratic in the curvature, such as R^2, $R_{\mu\nu}R^{\mu\nu}$, or $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$, and the Einstein equations are modified by higher derivative terms. The R^2 terms can be of fundamental origin or, as we will see later, can arise due to vacuum polarization effects.

The simplest action describing the dynamics of the electromagnetic field A_μ itself can be easily built out of the gauge invariant *field strength*

$$F_{\mu\nu} \equiv A_{\nu;\mu} - A_{\mu;\nu} = A_{\nu,\mu} - A_{\mu,\nu}, \tag{5.20}$$

and the result is

$$S\left[A_\mu\right] = -\frac{1}{16\pi} \int d^4x \sqrt{-g} g^{\alpha\beta} g^{\mu\nu} F_{\alpha\mu} F_{\beta\nu}. \tag{5.21}$$

It also describes the coupling of the electromagnetic field to gravity. Action (5.21) is conformally invariant: under the conformal transformation (5.6) the factors $\sqrt{-g}$ and $g^{\alpha\beta}$ are multiplied by Ω^4 and Ω^{-2} respectively, resulting in no net change in the action. Therefore the evolution of the electromagnetic field in conformally flat spacetimes is greatly simplified. In particular, the gravitational field in the Friedmann universe does not produce any photons.

5.5 Energy-momentum tensor

The total action describing gravity coupled to the matter fields ϕ_i can be written as

$$S\left[\phi_i, g_{\mu\nu}\right] = S^{\text{grav}}\left[g_{\mu\nu}\right] + S^m\left[\phi_i, g_{\mu\nu}\right].$$

The equations of motion for the gravitational field are obtained by varying this action with respect to the metric:

$$\frac{\delta S\left[\phi_i, g_{\mu\nu}\right]}{\delta g^{\alpha\beta}} = \frac{\delta}{\delta g^{\alpha\beta}} S^{\text{grav}}\left[g_{\mu\nu}\right] + \frac{\delta}{\delta g^{\alpha\beta}} S^m\left[\phi_i, g_{\mu\nu}\right] = 0. \tag{5.22}$$

These equations must coincide with the Einstein equations,

$$G_{\alpha\beta} \equiv R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = 8\pi G T_{\alpha\beta}, \tag{5.23}$$

where $T_{\alpha\beta}$ is the energy-momentum tensor of the matter fields ϕ_i. Taking into account (5.23) we find that the equations (5.22) and (5.23) are consistent only if

$$T_{\alpha\beta} = \frac{2}{\sqrt{-g}} \frac{\delta S^m}{\delta g^{\alpha\beta}}. \tag{5.24}$$

Thus equation (5.24) can be viewed as a *definition* of the energy-momentum tensor (EMT) for the matter fields. The resulting tensor $T_{\alpha\beta}$ is automatically symmetric and covariantly conserved,

$$T^{\alpha}_{\beta;\alpha} = 0. \tag{5.25}$$

Example 5.1 The energy-momentum tensor of the minimally coupled scalar field ϕ with the action (5.4) is

$$T_{\alpha\beta}(x) = \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\alpha\beta}(x)} = \phi_{,\alpha}\phi_{,\beta} - g_{\alpha\beta}\left[\frac{1}{2} g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - V(\phi)\right]. \tag{5.26}$$

Conservation of the EMT The covariant conservation of the EMT is a consequence of the invariance of the action with respect to general coordinate transformations. Considering an infinitesimal coordinate transformation

$$x^\alpha \to \tilde{x}^\alpha = x^\alpha + \xi^\alpha(x), \tag{5.27}$$

we find that the metric transforms as

$$g^{\alpha\beta}(x) \to \tilde{g}^{\alpha\beta}(x) = g^{\alpha\beta}(x) + \xi^{\alpha;\beta} + \xi^{\beta;\alpha} + O\left(|\xi|^2\right).$$

Note that $g^{\alpha\beta}(x)$ and $\tilde{g}^{\alpha\beta}(x)$ refer to different points of the manifold which have the same coordinate values in two different coordinate systems x and $\tilde{x}$. The specific form of the transformation law for the matter field,

$$\phi_i(x) \to \tilde{\phi}_i(x) = \phi_i(x) + \delta\phi_i(x),$$

depends on the type of the field. For example, for the real scalar field

$$\phi(x) \to \tilde{\phi}(x) = \phi(x) - \phi_{,\beta}\xi^{\beta}. \tag{5.28}$$

The action does not depend on the coordinate system and hence its variation under the coordinate transformation (5.27) must vanish, that is,

$$\delta S^m = \int \frac{\delta S^m}{\delta g^{\alpha\beta}(x)}\left(\xi^{\alpha;\beta} + \xi^{\beta;\alpha}\right) d^4x + \int \frac{\delta S^m}{\delta \phi_i(x)} \delta\phi_i(x) d^4x = 0. \tag{5.29}$$

Taking into account (5.24), one obtains

$$\begin{aligned} \int \frac{\delta S^m}{\delta g^{\alpha\beta}(x)}\left(\xi^{\alpha;\beta} + \xi^{\beta;\alpha}\right) d^4x &= \int T_{\alpha\beta}\xi^{\beta;\alpha}\sqrt{-g}d^4x \\ = \int \left[\left(T^{\alpha}_{\beta}\xi^{\beta}\right)_{;\alpha} - T^{\alpha}_{\beta;\alpha}\xi^{\beta}\right]\sqrt{-g}d^4x &= -\int T^{\alpha}_{\beta;\alpha}\xi^{\beta}\sqrt{-g}d^4x, \end{aligned} \tag{5.30}$$

where we have assumed that ξ^{α} vanishes sufficiently quickly at infinity. Considering the real scalar field ϕ and substituting (5.30) and (5.28) into (5.29), we find that $\delta S^m = 0$ only if

$$T^{\alpha}_{\beta;\alpha} + \frac{\delta S^m}{\delta\phi}\phi_{,\beta} = 0. \tag{5.31}$$

It follows that the energy-momentum tensor is covariantly conserved, $T^{\alpha}_{\beta;\alpha} = 0$, if the equations of motion

$$\frac{\delta S^m}{\delta\phi} = 0 \tag{5.32}$$

are satisfied.

Moreover, one can easily see from (5.31) that the covariant conservation of T^{α}_{β} is equivalent to the equations of motion (5.32). The Einstein tensor $G_{\alpha\beta}$ defined in (5.23) satisfies the Bianchi identities $G^{\alpha}_{\beta;\alpha} = 0$ and hence the conservation of the energy-momentum tensor is incorporated in the Einstein equations. Therefore the equations of motion for matter do not need to be postulated separately.

If the action for the matter field is invariant with respect to the conformal transformations (5.6) then the trace of the corresponding EMT vanishes. In fact, considering an infinitesimal conformal variation of the metric,

$$\delta g^{\alpha\beta} = g^{\alpha\beta}\delta\Theta, \tag{5.33}$$

where $\delta\Theta(x)$ is an arbitrary function, we find that

$$\delta S^m = \int \frac{\delta S^m}{\delta g^{\alpha\beta}} \delta g^{\alpha\beta} d^4x = \int \frac{\sqrt{-g}}{2} T_{\alpha\beta} g^{\alpha\beta} \delta\Theta d^4x = \int \frac{\sqrt{-g}}{2} T^{\alpha}_{\alpha} \delta\Theta d^4x,$$

and hence $T^{\alpha}_{\alpha} = 0$ if S^m is invariant with respect to transformation (5.33). We will show in the second part of the book that the vacuum polarization effects generally spoil conformal invariance of the original classical theory and generate a nonzero value of T^{α}_{α}. This phenomenon is called the *trace anomaly*.