3

Driven harmonic oscillator

Summary Quantization of harmonic oscillator driven by external classical force. "In" and "out" states. Matrix elements and Green's functions.

A quantum harmonic oscillator driven by an external classical force is a simple physical system which allows us to introduce several important concepts, such as Green's functions, "in" and "out" states, and to formulate the problem of particle production.

3.1 Quantizing an oscillator

A harmonic oscillator driven by a given external force $J(t)$ satisfies the classical equation of motion

$$\ddot{q} = -\omega^2 q + J(t)$$

which follows from the Lagrangian

$$L(t, q, \dot{q}) = \frac{1}{2}\dot{q}^2 - \frac{1}{2}\omega^2 q^2 + J(t)q.$$

The corresponding Hamiltonian is

$$H(p, q) = \frac{p^2}{2} + \frac{\omega^2 q^2}{2} - J(t)q, \tag{3.1}$$

and the Hamilton equations of motion are

$$\dot{q} = p, \quad \dot{p} = -\omega^2 q + J(t). \tag{3.2}$$

Note that the Hamiltonian depends explicitly on the time t and therefore the energy of the harmonic oscillator is not conserved.

In quantum theory the coordinate q and the momentum p become the operators $\hat{q}(t)$ and $\hat{p}(t)$ satisfying the equal time commutation relation $[\hat{q}, \hat{p}] = i$. (From now on, we use the units where $\hbar = 1$.) In the Heisenberg picture these operators satisfy the Heisenberg equations which are obtained by replacing q and p in (3.2) by the corresponding operators:

$$\frac{d\hat{q}}{dt} = \hat{p}, \quad \frac{d\hat{p}}{dt} = -\omega^2 \hat{q} + J(t). \tag{3.3}$$

It is convenient to introduce two Hermitian conjugated operators $\hat{a}^-(t)$ and $\hat{a}^+(t)$ instead of $\hat{p}(t)$ and $\hat{q}(t)$:

$$\hat{a}^-(t) \equiv \sqrt{\frac{\omega}{2}} \left[\hat{q}(t) + \frac{i}{\omega}\hat{p}(t) \right], \quad \hat{a}^+(t) \equiv \sqrt{\frac{\omega}{2}} \left[\hat{q}(t) - \frac{i}{\omega}\hat{p}(t) \right].$$

These operators, called the *annihilation* and *creation* operators respectively, satisfy the commutation relation

$$\left[\hat{a}^-(t), \hat{a}^+(t) \right] = 1 \tag{3.4}$$

at every moment of time t.

Exercise 3.1
Using the commutation relation $[\hat{q}, \hat{p}] = i$, verify that $[\hat{a}^-(t), \hat{a}^+(t)] = 1$.

The equations of motion for the operators $\hat{a}^{\mp}(t)$ follow immediately from (3.3):

$$\frac{d}{dt}\hat{a}^- = -i\omega\hat{a}^- + \frac{i}{\sqrt{2\omega}} J(t), \tag{3.5}$$

$$\frac{d}{dt}\hat{a}^+ = i\omega\hat{a}^+ - \frac{i}{\sqrt{2\omega}} J(t). \tag{3.6}$$

They are readily integrated to give

$$\hat{a}^-(t) = \left[\hat{a}^-_{\text{in}} + \frac{i}{\sqrt{2\omega}} \int_0^t e^{i\omega t'} J(t') dt' \right] e^{-i\omega t}, \tag{3.7}$$

$$\hat{a}^+(t) = \left[\hat{a}^+_{\text{in}} - \frac{i}{\sqrt{2\omega}} \int_0^t e^{-i\omega t'} J(t') dt' \right] e^{i\omega t}, \tag{3.8}$$

where $\hat{a}^{\mp}_{in} = a^{\mp}(t = 0)$ are the operator-valued constants of integration.

Exercise 3.2
Derive solutions (3.7) and (3.8).

Substituting

$$\hat{q} = \frac{1}{\sqrt{2\omega}} \left(\hat{a}^- + \hat{a}^+ \right), \quad \hat{p} = \frac{\sqrt{\omega}}{i\sqrt{2}} \left(\hat{a}^- - \hat{a}^+ \right) \tag{3.9}$$

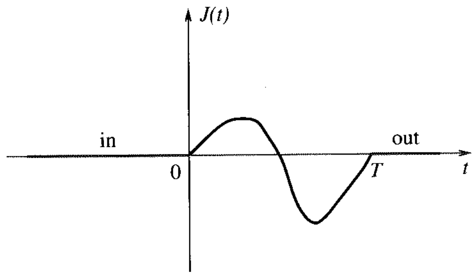

Fig. 3.1 The external force $J(t)$ and the "in"/"out" regions.

into (3.1), we find the following expression for the Hamiltonian in terms of the creation and annihilation operators $\hat{a}^{\pm}$,

$$\hat{H} = \frac{\omega}{2}\left(\hat{a}^+\hat{a}^- + \hat{a}^-\hat{a}^+\right) - \frac{\hat{a}^+ + \hat{a}^-}{\sqrt{2\omega}}J(t)$$
$$= \frac{\omega}{2}\left(2\hat{a}^+\hat{a}^- + 1\right) - \frac{\hat{a}^+ + \hat{a}^-}{\sqrt{2\omega}}J(t). \tag{3.10}$$

3.2 The "in" and "out" states

To simplify the calculations, we assume that the external force $J(t)$ is nonzero only during a time interval $T > t > 0$. The regions $t \leq 0$ and $t \geq T$, where the oscillator is unperturbed, are called the "in" and "out" regions respectively (see Fig. 3.1). Our purpose is to determine the relation between the states of the oscillator in these regions.

It follows from (3.7) and (3.8) that

$$\hat{a}^-(t) = \hat{a}^-_{\text{in}} e^{-i\omega t}, \quad \hat{a}^+(t) = \hat{a}^+_{\text{in}} e^{i\omega t} \tag{3.11}$$

in the "in" region. Correspondingly in the "out" region we have

$$\hat{a}^-(t) = \hat{a}^-_{\text{out}} e^{-i\omega t}, \quad \hat{a}^+(t) = \hat{a}^+_{\text{out}} e^{i\omega t} \tag{3.12}$$

where

$$\hat{a}^-_{\text{out}} \equiv \hat{a}^-_{\text{in}} + \frac{i}{\sqrt{2\omega}}\int_0^T e^{i\omega t'} J(t')dt' \equiv \hat{a}^-_{\text{in}} + J_0, \quad \hat{a}^+_{\text{out}} = \hat{a}^+_{\text{in}} + J_0^*. \tag{3.13}$$

It is obvious that both pairs of the time-independent operators $\hat{a}^{\pm}_{\text{in}}$ and $\hat{a}^{\pm}_{\text{out}}$ satisfy the commutation relation $\left[\hat{a}^-, \hat{a}^+\right] = 1$. Substituting (3.11) and (3.12) into (3.10), we find that the Hamiltonian

$$\hat{H} = \begin{cases} \omega\left(\hat{a}^+_{\text{in}}\hat{a}^-_{\text{in}} + \frac{1}{2}\right) & \text{for } t \le 0, \\ \omega\left(\hat{a}^+_{\text{out}}\hat{a}^-_{\text{out}} + \frac{1}{2}\right) & \text{for } t \ge T, \end{cases} \tag{3.14}$$

does not depend on time in the "in" and "out" regions.

Hilbert space It is well known how to construct the Hilbert space of quantum states for the unperturbed oscillator. We *assume* the existence of the unique normalized vector $|0\rangle$ for which $\hat{a}^- |0\rangle = 0$. This vector corresponds to the ground (vacuum) state. One can then prove that the orthogonal normalized vectors

$$|n\rangle = \frac{1}{\sqrt{n!}}(\hat{a}^+)^n |0\rangle, \tag{3.15}$$

describing the excited states for $n \ge 1$, form a complete basis in the Hilbert space. In other words, all possible quantum states of the oscillator are of the form

$$|\psi\rangle = \sum_{n=0}^{\infty} \psi_n |n\rangle, \quad \sum_{n=0}^{\infty} |\psi_n|^2 = 1. \tag{3.16}$$

This is a standard result and we omit its proof. Details can be found e.g. in the book by P. A. M. Dirac, *Principles of Quantum Mechanics* (Oxford, 1948). Ultimately, the agreement between the theory and experiment determines whether a particular Hilbert space is suitable for describing a physical system; for a harmonic oscillator, the space spanned by the orthonormal basis $\{|n\rangle\}$, where $n = 0, 1, \ldots$, is adequate.

In the case under consideration there are two regions where the external classical force vanishes. Therefore, two annihilation operators, $\hat{a}^-_{\text{in}}$ and $\hat{a}^-_{\text{out}}$, define two different vacuum states – the "in" vacuum $|0_{\text{in}}\rangle$ and the "out" vacuum $|0_{\text{out}}\rangle$:

$$\hat{a}^-_{\text{in}} |0_{\text{in}}\rangle = 0, \quad \hat{a}^-_{\text{out}} |0_{\text{out}}\rangle = 0.$$

It follows from (3.14) that the vectors $|0_{\text{in}}\rangle$ and $|0_{\text{out}}\rangle$ are the lowest-energy states for $t \le 0$ and for $t \ge T$ respectively. One can easily see that the vectors $|0_{\text{in}}\rangle$ and $|0_{\text{out}}\rangle$ are different: The state $|0_{\text{out}}\rangle$ is an eigenstate of the operator $\hat{a}^-_{\text{out}}$ with zero eigenvalue, while

$$\hat{a}^-_{\text{out}} |0_{\text{in}}\rangle = (\hat{a}^-_{\text{in}} + J_0) |0_{\text{in}}\rangle = J_0 |0_{\text{in}}\rangle,$$

that is, the vector $|0_{\text{in}}\rangle$ is an eigenstate of the same operator with the eigenvalue J_0. (We recall that the eigenstates of the annihilation operator with nonzero eigenvalues are called the *coherent states*.) Conversely, $\hat{a}^-_{\text{in}} |0_{\text{in}}\rangle = 0$ and $\hat{a}^-_{\text{in}} |0_{\text{out}}\rangle = -J_0 |0_{\text{out}}\rangle$.

Using the creation operators $\hat{a}_{\text{in}}^{+}$ and $\hat{a}_{\text{out}}^{+}$, we can build two complete orthonormal sets of excited states,

$$|n_{\text{in}}\rangle = \frac{1}{\sqrt{n!}} \left(\hat{a}_{\text{in}}^{+}\right)^n |0_{\text{in}}\rangle, \quad |n_{\text{out}}\rangle = \frac{1}{\sqrt{n!}} \left(\hat{a}_{\text{out}}^{+}\right)^n |0_{\text{out}}\rangle, \quad n = 1, 2, \ldots$$

It is easy to verify that

$$\hat{H}(t)\,|n_{\text{in}}\rangle = \omega\left(n + \frac{1}{2}\right)|n_{\text{in}}\rangle, \quad t \le 0;$$

$$\hat{H}(t)\,|n_{\text{out}}\rangle = \omega\left(n + \frac{1}{2}\right)|n_{\text{out}}\rangle, \quad t \ge T.$$

Hence, the vectors $|n_{\text{in}}\rangle$ are the eigenstates of the Hamiltonian (3.14) for $t \le 0$ (but not for $t \ge T$), and $|n_{\text{out}}\rangle$ are its eigenstates for $t \ge T$. For this reason it is natural to interpret $|n_{\text{in}}\rangle$ as "n-particle states" of the oscillator for $t \le 0$, while for $t \ge T$ the n-particle states are $|n_{\text{out}}\rangle$.

Remark: interpretation of the "in" and "out" states We work in the Heisenberg picture where quantum states are time-independent and operators change in time. In this picture the physical interpretation of a constant state vector $|\psi\rangle$ depends on time. For example, we have found that the vector $|0_{\text{in}}\rangle$ is no longer the lowest-energy state for $t \ge T$. This happens because the energy of the system changes due to the external force $J(t)$. In the absence of this force, $\hat{a}_{\text{in}}^{-} = \hat{a}_{\text{out}}^{-}$ and the state $|0_{\text{in}}\rangle$ describes the vacuum state at all times.

Relation between "in" and "out" states Both sets of states $\{|n_{\text{in}}\rangle\}$ and $\{|n_{\text{out}}\rangle\}$ form separately a complete basis in the Hilbert space. Therefore the vector $|0_{\text{in}}\rangle$ can be written as a linear combination of the "out" states,

$$|0_{in}\rangle = \sum_{n=0}^{\infty} \Lambda_n\, |n_{\text{out}}\rangle. \tag{3.17}$$

The coefficients Λ_n satisfy the recurrence relation

$$\Lambda_{n+1} = \frac{J_0}{\sqrt{n+1}} \Lambda_n. \tag{3.18}$$

Exercise 3.3
Derive the recurrence relation (3.18) from (3.13).

It is easy to see that the solution of (3.18) is

$$\Lambda_n = \frac{J_0^n}{\sqrt{n!}} \Lambda_0.$$

The constant Λ_0 is fixed by the normalization condition $\langle 0_{\text{in}} | 0_{\text{in}} \rangle = 1$. To compute Λ_0, we consider the normalization

$$\langle 0_{\text{in}} | 0_{\text{in}} \rangle = \sum_{n=0}^{\infty} |\Lambda_n|^2 = |\Lambda_0|^2 \sum_{n=0}^{\infty} \frac{|J_0|^{2n}}{n!} = |\Lambda_0|^2 e^{|J_0|^2} = 1$$

and hence

$$|\Lambda_0| = \exp\left[-\frac{1}{2}|J_0|^2\right].$$

The unimportant phase of Λ_0 remains undetermined.

Thus, the state $|0_{\text{in}}\rangle$ can be expressed in terms of the "out" states as

$$|0_{\text{in}}\rangle = \exp\left[-\frac{1}{2}|J_0|^2\right] \sum_{n=0}^{\infty} \frac{J_0^n}{\sqrt{n!}} |n_{\text{out}}\rangle, \tag{3.19}$$

or, equivalently,

$$|0_{\text{in}}\rangle = \exp\left[-\frac{1}{2}|J_0|^2 + J_0 \hat{a}^+_{\text{out}}\right] |0_{\text{out}}\rangle.$$

Hence for $t > T$ the state vector $|0_{\text{in}}\rangle$ describes the coherent state of the harmonic oscillator, which is an eigenstate of $\hat{a}^-_{\text{out}}$ with eigenvalue J_0.

It follows from (3.19) that the initial vacuum state is a superposition of excited states for $t > T$. In particular, the probability of detecting the oscillator in an excited state n is

$$|\Lambda_n|^2 = e^{-|J_0|^2} \frac{|J_0|^{2n}}{n!}.$$

If we interpret a state with the occupation number n as describing n particles, then one concludes that the presence of the external force $J(t)$ leads to the particle production.

3.3 Matrix elements and Green's functions

The experimentally measurable quantities are expectation values of various Hermitian operators such as $\langle 0_{\text{in}} | \hat{q}(t) | 0_{\text{in}} \rangle$. Unlike the expectation values, "in-out" matrix elements, e.g. $\langle 0_{\text{out}} | \hat{q}(t) | 0_{\text{in}} \rangle$, cannot be directly measured and they are in general complex numbers. However, as we shall see in Chapter 12, such matrix elements are sometimes useful in intermediate calculations. Therefore we shall now calculate the expectation values and the matrix elements for various operators for $t \leq 0$ (the "in" region) and $t \geq T$ (the "out" region).

Example 3.1 First we compute the expectation value of the Hamiltonian $\hat{H}(t)$ for the "in" vacuum state $|0_{in}\rangle$. It follows from (3.14) that

$$\langle 0_{in}|\, \hat{H}(t)\, |0_{in}\rangle = \frac{\omega}{2}$$

for $t \leq 0$ and

$$\langle 0_{in}|\, \hat{H}(t)\, |0_{in}\rangle = \langle 0_{in}|\, \omega \left(\frac{1}{2} + \hat{a}^{+}_{out}\hat{a}^{-}_{out}\right) |0_{in}\rangle = \left(\frac{1}{2} + |J_0|^2\right) \omega$$

for $t \geq T$, where the relations in (3.13) have been used to express $\hat{a}^{\pm}_{out}$ in terms of $\hat{a}^{\pm}_{in}$.

It is now apparent that the energy of the oscillator becomes larger than the zero-point energy $\frac{1}{2}\omega$ after applying the force $J(t)$. The constant $|J_0|^2$ is expressed in terms of $J(t)$ as

$$|J_0|^2 = \frac{1}{2\omega} \int_0^T \int_0^T dt_1 dt_2 e^{i\omega(t_1 - t_2)} J(t_1)\, J(t_2)\,.$$

Example 3.2 The occupation number operator, defined as

$$\hat{N}(t) \equiv \hat{a}^{+}(t)\hat{a}^{-}(t) = \begin{cases} \hat{a}^{+}_{in}\hat{a}^{-}_{in} & \text{for} \quad t \leq 0, \\ \hat{a}^{+}_{out}\hat{a}^{-}_{out} & \text{for} \quad t \geq T, \end{cases}$$

has the expectation value

$$\langle 0_{in}|\, \hat{N}(t)\, |0_{in}\rangle = \begin{cases} 0 & \text{for} \quad t \leq 0, \\ |J_0|^2 & \text{for} \quad t \geq T. \end{cases} \tag{3.20}$$

The in-out matrix element of $\hat{N}(t)$ is

$$\langle 0_{out}|\, \hat{N}(t)\, |0_{in}\rangle = 0$$

for $t \leq 0$ and $t \geq T$.

Example 3.3 The expectation value of the position operator,

$$\hat{q}(t) = \frac{1}{\sqrt{2\omega}} \left(\hat{a}^{-}(t) + \hat{a}^{+}(t)\right), \tag{3.21}$$

is equal to zero,

$$\langle 0_{in}|\, \hat{q}(t \leq 0)\, |0_{in}\rangle = 0, \tag{3.22}$$

for $t \leq T$. It follows from (3.12) and (3.13) that

$$\hat{a}^-(t \geq T) = \hat{a}^-_{\text{out}} e^{-i\omega t} = (\hat{a}^-_{\text{in}} + J_0)\, e^{-i\omega t},$$
$$\hat{a}^+(t \geq T) = (\hat{a}^+_{\text{in}} + J_0^*)\, e^{i\omega t},$$

and therefore

$$\langle 0_{\text{in}} |\, \hat{q}(t)\, | 0_{\text{in}} \rangle = \frac{1}{\sqrt{2\omega}} \left(J_0 e^{-i\omega t} + J_0^* e^{i\omega t} \right) = \int_0^T \frac{\sin \omega (t - t')}{\omega} J(t') dt' \qquad (3.23)$$

for $t \geq T$.

Introducing the *retarded Green's function* for the harmonic oscillator,

$$G_{\text{ret}}(t, t') \equiv \frac{\sin \omega (t - t')}{\omega} \theta(t - t'), \qquad (3.24)$$

results (3.22), (3.23) can be rewritten as

$$q(t) = \int_{-\infty}^{+\infty} J(t') G_{\text{ret}}(t, t') dt'. \qquad (3.25)$$

Example 3.4 The in-out matrix element of the position operator $\hat{q}$ is

$$\frac{\langle 0_{\text{out}} |\, \hat{q}(t \leq 0)\, | 0_{\text{in}} \rangle}{\langle 0_{\text{out}} | 0_{\text{in}} \rangle} = \frac{e^{-i\omega t}}{\sqrt{2\omega}} \frac{\langle 0_{\text{out}} |\, \hat{a}^+_{\text{in}}\, | 0_{\text{in}} \rangle}{\langle 0_{\text{out}} | 0_{\text{in}} \rangle} = -J_0 \frac{e^{-i\omega t}}{\sqrt{2\omega}},$$
$$\frac{\langle 0_{\text{out}} |\, \hat{q}(t \geq T)\, | 0_{\text{in}} \rangle}{\langle 0_{\text{out}} | 0_{\text{in}} \rangle} = \frac{e^{-i\omega t}}{\sqrt{2\omega}} \frac{\langle 0_{\text{out}} |\, \hat{a}^-_{\text{out}}\, | 0_{\text{in}} \rangle}{\langle 0_{\text{out}} | 0_{\text{in}} \rangle} = J_0 \frac{e^{-i\omega t}}{\sqrt{2\omega}}. \qquad (3.26)$$

In general, these matrix elements are complex numbers. Noting that

$$\frac{1}{\sqrt{2\omega}} J_0 e^{-i\omega t} = \frac{i}{2\omega} \int_0^T e^{-i\omega(t - t')} J(t') dt',$$

and introducing the *Feynman Green's function*

$$G_{\text{F}}(t, t') \equiv \frac{i e^{-i\omega |t - t'|}}{2\omega} \qquad (3.27)$$

result (3.26) can be rewritten as

$$\frac{\langle 0_{\text{out}} |\, \hat{q}(t)\, | 0_{\text{in}} \rangle}{\langle 0_{\text{out}} | 0_{\text{in}} \rangle} = \int_{-\infty}^{+\infty} J(t') G_{\text{F}}(t, t') dt'. \qquad (3.28)$$

Other matrix elements, such as $\langle 0_{\text{in}} |\, \hat{q}(t_1)\, \hat{q}(t_2)\, | 0_{\text{in}} \rangle$, can be calculated similarly.

Exercise 3.4

Verify that for $t_1,\ t_2 \geq T$

$$\langle 0_{\text{in}}|\,\hat{q}(t_2)\,\hat{q}(t_1)\,|0_{\text{in}}\rangle = \frac{1}{2\omega}e^{-i\omega(t_2-t_1)} + \int_0^T dt_1' \int_0^T dt_2' J(t_1')\,J(t_2')\,G_{\text{ret}}(t_1,t_1')\,G_{\text{ret}}(t_2,t_2')$$

and

$$\frac{\langle 0_{\text{out}}|\,\hat{q}(t_2)\,\hat{q}(t_1)\,|0_{\text{in}}\rangle}{\langle 0_{\text{out}}|0_{\text{in}}\rangle} = \frac{1}{2\omega}e^{-i\omega(t_2-t_1)} + \int_0^T dt_1' \int_0^T dt_2' J(t_1')\,J(t_2')\,G_{\text{F}}(t_1,t_1')\,G_{\text{F}}(t_2,t_2').$$