10

The Casimir effect

Summary Zero-point energy in the presence of boundaries. Regularization and renormalization. Casimir force.

The Casimir effect is an experimentally verified prediction of quantum field theory. It is manifested by a force of attraction between two *uncharged* conducting plates in a vacuum. This force cannot be explained except by considering the vacuum fluctuations of the quantized electromagnetic field. The presence of the conducting plates makes the electromagnetic field vanish on the surfaces of the plates, causing a finite shift ΔE of the zero-point energy. This shift depends on the distance L between the plates, and as a result there arises the *Casimir force*:

$$F(L) = -\frac{d}{dL}\Delta E(L).$$

This theoretical prediction has been verified experimentally.[1]

10.1 Vacuum energy between plates

A realistic description of the Casimir effect requires quantization of the electromagnetic field in the presence of conducting plates. To simplify the calculations, we consider a two-dimensional massless scalar field $\phi(t, x)$ between two plates at $x = 0$ and $x = L$, imposing the boundary conditions

$$\phi(t, x)|_{x=0} = \phi(t, x)|_{x=L} = 0,$$

which are supposed to be due to the presence of the plates. With these boundary conditions the general solution of the classical equation of motion,

$$\partial_t^2\phi - \partial_x^2\phi = 0,$$

[1] For example, a recent measurement of the Casimir force to 1% precision is described in: U. Mohideen and A. Roy, *Phys. Rev. Lett.* **81** (1998), 4549.

becomes

$$\phi(t,x) = \sum_{n=-\infty}^{\infty} \left(A_n e^{-i\omega_n t} + B_n e^{i\omega_n t}\right) \sin \omega_n x, \quad \omega_n \equiv \frac{|n|\,\pi}{L}. \tag{10.1}$$

In quantum theory only the modes present in (10.1) "survive" in the expansion of the field operator $\hat{\phi}$, which becomes

$$\hat{\phi}(t,x) = \sqrt{\frac{1}{L}} \sum_{n=1}^{\infty} \frac{\sin \omega_n x}{\sqrt{\omega_n}} \left[\hat{a}_n^- e^{-i\omega_n t} + \hat{a}_n^+ e^{i\omega_n t}\right]. \tag{10.2}$$

The resulting zero-point energy per unit length between the plates is then

$$\varepsilon_0 \equiv \frac{1}{L} \langle 0| \hat{H} |0\rangle = \frac{1}{2L} \sum_k \omega_k = \frac{\pi}{2L^2} \sum_{n=1}^{\infty} n. \tag{10.3}$$

Exercise 10.1

(a) Show that the mode expansion (10.2) yields the standard commutation relations $[\hat{a}_m^-, \hat{a}_n^+] = \delta_{mn}$.

(b) Derive (10.3).

Hint: Use the identities which hold for integers m, n:

$$\int_0^L dx \sin \frac{m\pi x}{L} \sin \frac{n\pi x}{L} = \int_0^L dx \cos \frac{m\pi x}{L} \cos \frac{n\pi x}{L} = \frac{L}{2} \delta_{mn}. \tag{10.4}$$

10.2 Regularization and renormalization

The zero-point energy density ε_0 is divergent and must be first regularized and then renormalized. A *regularization* means introducing an extra parameter (cutoff scale) into the theory to make the divergent quantity finite unless that parameter is set either to zero or infinity depending on the concrete regularization procedure used. Usually there exist different possible ways to regularize the divergent quantities. However, different regularization procedures (fortunately) lead to the same final physical results. After a regularization, one obtains an asymptotic expansion of the regularized divergent quantity at small (or large) values of the cutoff. This asymptotic expansion may contain divergent powers and logarithms of the cutoff scale, as well as finite terms. *Renormalization* gives a physical justification for removing the divergent terms and leaves us with finite contributions responsible for physical effects.

We shall now apply this procedure to (10.3). As a first step, we replace ε_0 by the regularized quantity

$$\varepsilon_0 (L; \alpha) = \frac{\pi}{2L^2} \sum_{n=1}^{\infty} n \exp\left[-\frac{n\alpha}{L}\right], \tag{10.5}$$

where α is the cutoff parameter. It is easy to see that the series in (10.5) converges for $\alpha > 0$, while the original divergent expression is recovered in the limit $\alpha \to 0$.

Remark We regularize the series by $\exp(-n\alpha/L)$ and not by $\exp(-n\alpha)$ or $\exp(-nL\alpha)$. The motivation is that the physically significant quantity is $\omega_n = \pi n/L$, therefore the cutoff factor should be a function of ω_n.

A straightforward calculation gives

$$\varepsilon_0(L;\alpha) = -\frac{\pi}{2L}\frac{\partial}{\partial\alpha}\sum_{n=1}^{\infty}\exp\left[-\frac{n\alpha}{L}\right] = \frac{\pi}{2L^2}\frac{\exp\left(-\frac{\alpha}{L}\right)}{\left[1-\exp\left(-\frac{\alpha}{L}\right)\right]^2}.$$

For small α this expression can be expanded in a Laurent series,

$$\varepsilon_0(L;\alpha) = \frac{\pi}{8L^2}\frac{1}{\sinh^2\frac{\alpha}{2L}} = \frac{\pi}{2\alpha^2} - \frac{\pi}{24L^2} + \frac{1}{L^2}O\left(\frac{\alpha^2}{L^2}\right). \tag{10.6}$$

As $\alpha \to 0$, the first term here diverges as α^{-2}, the second term remains finite and further terms vanish. The crucial fact is that the singular term does not depend on the distance L between the plates and can be interpreted as the energy density of the zero-point fluctuations in Minkowski spacetime without boundaries. This zero-point energy density can be thought of as the limit of $\varepsilon_0(L)$ as $L \to \infty$ and, as it is clear from (10.6), must be exactly equal to the first divergent term in (10.6). On the other hand, we have agreed to ignore an infinite energy of zero-point fluctuations in Minkowski space, assuming that this energy does not contribute to any physically relevant quantities. This is the physical justification for omitting the divergent contribution to (10.6).

Subtracting from (10.6) the vacuum energy density and removing the cutoff (taking the limit $\alpha \to 0$), we obtain

$$\Delta\varepsilon_{\text{ren}}(L) = \lim_{\alpha\to 0}\left[\varepsilon_0(L;\alpha) - \lim_{L\to\infty}\varepsilon_0(L;\alpha)\right] = -\frac{\pi}{24L^2}. \tag{10.7}$$

After we have decided to fix the normalization point and to attribute a "zero" energy to vacuum fluctuations in Minkowski spacetime, there remains no more freedom to renormalize the finite shift of the energy density due to the presence of the plates. Therefore this energy shift is physical. The corresponding Casimir force between the plates is

$$F = -\frac{d}{dL}\Delta E = -\frac{d}{dL}(L\Delta\varepsilon_{\text{ren}}) = -\frac{\pi}{24L^2},$$

where the negative sign tells us that the plates are pulled toward each other.

A similar calculation in four-dimensional spacetime gives the Casimir force per unit area between two uncharged parallel plates as

$$f = -\frac{\pi^2}{240} L^{-4}.$$

Remark: negative energy Note that the shift of the energy density in (10.7) is negative. Quantum field theory generally admits quantum states with a negative expectation value of energy.

Riemann's zeta function regularization An elegant and quick way to calculate the finite energy shift due to the plates is with the help of Riemann's zeta (ζ) function defined by the series

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}, \tag{10.8}$$

which converges for real $x > 1$. An analytic continuation extends this function to all (complex) x, except $x = 1$ where $\zeta(x)$ has a pole.

The divergent sum $\sum_{n=1}^{\infty} n$ appearing in (10.3) is *formally* equivalent to the series for $\zeta(x)$ with $x = -1$. The ζ function obtained via analytic continuation is, however, finite at $x = -1$ and is equal to[2]

$$\zeta(-1) = -\frac{1}{12}.$$

This motivates us to replace the divergent sum $\sum_{n=1}^{\infty} n$ in (10.3) by the number $-\frac{1}{12}$. After this substitution, we immediately obtain the result (10.7).

At a first glance, this procedure may appear miraculous and lacking of physical explanation of neglecting divergences, unlike the straightforward renormalization approach. However, it has been verified in many cases that the results obtained using the ζ function method are in agreement with more direct renormalization procedures.

[2] This result requires a complicated proof. See e.g. H. Bateman and A. Erdelyi, *Higher Transcendental Functions*, vol. 1 (McGraw-Hill, New York, 1953).