Today we study tensor analysis from a general standpoint (working on an arbitrary space with arbitrary coordinates), and then specialize to the case of space-time in special relativity with Lorentz coordinates. This will explain the reasons for many of the features of tensor analysis in special relativity, for example, the positions of the indices and what that means. Also, we will need the general tensor analysis later when we look at general relativity.

To begin, a space upon which coordinates can be imposed is called a manifold. The dimension of the manifold is the number of coordinates. For example, ordinary (physical) space is usually modeled as $\mathbb{R}^3$ in mathematical notation, on which we typically use coordinates (xyz) or $(r\theta\phi)$ or others. Space-time is modeled in special relativity as $\mathbb{R}^4$, where we usually use coordinates $(txyz) = x^\mu$, $\mu = 0,1,2,3$. An ordinary (2-dimensional) sphere is denoted S^2, and can be seen as a subset (a submanifold) of $\mathbb{R}^3$. Coordinates on S^2 may be (θ, ϕ). In classical mechanics, the configuration space is a manifold. In physics courses people often talk about configuration space as if it were the same as physical space, but that's only true for a single particle moving in 3 dimensions. More generally, config. space is an abstract space in which a single point gives all the information necessary to specify the positions of all the particles. For example, the orientation of a rigid body is specified by the Euler angles $(\alpha\beta\gamma)$ that give the rotation that maps a standard orientation of the rigid body into the actual one. So the configuration space of the rigid body with one point fixed is "Euler angle space", that is, the group manifold $SO(3)$ upon which the Euler angles are coordinates.

To be general, let x^i, $i = 1, \dots, n$ be coordinates on a manifold M, so $\dim M = n$. The superscript position for i is conventional.

Coordinates on a manifold are highly arbitrary, so we need to see what happens when we change them. Coordinates are really a subjective element when trying to describe physics, so the true physics must be independent of the coordinates used. Let x^i be an "old" coordinate system and x'^i a "new". Since coordinates are supposed to provide a unique labelling of points, the new coordinates must be functions of the old,

$$x'^i = x'^i(x)$$

means all the old coordinates, $(x^1, x^2, \ldots, x^n)$

Also, these functions must be invertible,

$$x^i = x^i(x').$$

Notice that we allow the coordinate transformation to be <u>nonlinear</u>. This is necessary on spaces that are not flat.

This means that the Jacobian matrix

$$J^i_{\cdot j} = \frac{\partial x'^i}{\partial x^j}$$

is invertible. It's inverse is the inverse Jacobian,

$$(J^{-1})^i_{\cdot j} = \frac{\partial x^i}{\partial x'^j},$$

Summation convention

since by the chain rule,

$$\frac{\partial x'^i}{\partial x'^j} = \frac{\partial x'^i}{\partial x^k}\frac{\partial x^k}{\partial x'^j} = \delta^i_j = J^i_{\cdot k}(J^{-1})^k_{\cdot j}$$

The positions of indices on $J^i_{\cdot j}$ are conventional, but will be explained in a moment.

Now we make some definitions. A qty f that has the same value in all coordinate systems,

$$f(x) = f(x')$$

is said to be a <u>scalar</u>.

The formula $f(x) = f(x')$ would be considered abuse of notation in mathematics, but it means that the value of f depends on the point at which we evaluate it, and not on the coordinates used to label that point. Think of the temperature in the room. Another

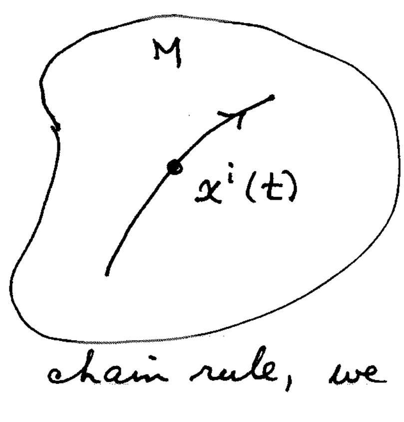
Def. A set of n quantities A^i, $i = 1, \dots, n$ such that

$$A'^i = \frac{\partial x'^i}{\partial x^j} A^j$$

is said to be (or to "transform as") a contravariant vector.

The idea here is that we have some mathematical or physical rule to compute the components of A^i in all coordinates. Then if the values of A^i in different coordinate systems are related by the formula above, we say that A^i is a contravariant vector.

Here is an example of a mathematical rule that leads to a contravariant vector. Suppose we have a curve on our manifold M parameterized by a parameter t. For example, it could be the orbit of a classical system in configuration space. The curve is described in old coordinates by $x^i(t)$, or in the new coordinates, $x'^i(t)$. Then by the chain rule, we have



$$\frac{dx'^i}{dt} = \frac{\partial x'^i}{\partial x^j} \frac{dx^j}{dt},$$

so if we write

$$v^i = \frac{dx^i}{dt}, \qquad v'^i = \frac{dx'^i}{dt},$$

then

$$v'^i = \frac{\partial x'^i}{\partial x^j} v^j ,$$

Call it the "velocity" vector.

and v^i forms a contravariant vector.

For example, for a particle moving in 3D space in rectangular coordinates, $v^i = (\dot{x}, \dot{y}, \dot{z})$, so the contravariant components are the same as the usual ones with respect to the basis $\hat{x}, \hat{y}, \hat{z}$, since

$$\vec{v} = \frac{d\vec{x}}{dt} = \dot{x}\,\hat{x} + \dot{y}\,\hat{y} + \dot{z}\,\hat{z}.$$

But in spherical components $(r\ \theta\ \phi)$ we have $v^i = (\dot{r}, \dot{\theta}, \dot{\phi})$. These are not the same as the components of $\vec{v}$ w.r.t. the basis $(\hat{r}, \hat{\theta}, \hat{\phi})$, since

$$\vec{v} = \frac{d\vec{x}}{dt} = \dot{r}\,\hat{r} + r\dot{\theta}\,\hat{\theta} + r\sin\theta\,\dot{\phi}\,\hat{\phi}.$$

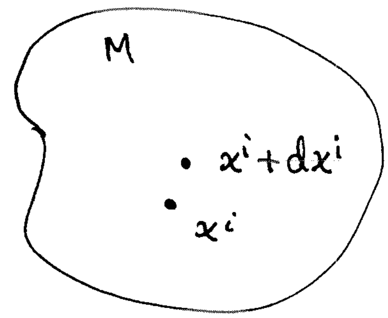
For a physical example of a rule giving the components of a vector in different coordinate systems, we can measure the energy and momentum of a particle in different frames, giving us

$$p^\mu = \begin{pmatrix} E \\ \vec{p} \end{pmatrix} \quad \text{or} \quad p'^\mu = \begin{pmatrix} E' \\ \vec{p}' \end{pmatrix}.$$

As pointed out in class, these measurements can be made without any knowledge of relativity theory, i.e. we can base the measurements on our knowledge of energy and momentum for nonrelativistic systems. If we do this, we will find that p^μ transforms as a 4-contravariant vector under Lorentz transformations. This can be taken as an experimental fact.

Not only is $v^i = \frac{dx^i}{dt}$ a contravariant vector, but so also are the coordinate differentials connecting any 2 points on M:

$$dx'^i = \frac{\partial x'^i}{\partial x^j} dx^j$$



by the chain rule. Of course we get the velocity again if we just divide by dt.

It's conventional to put an <u>upper</u> index on a contravariant vector. It makes a statement about how that vector transforms under a coordinate transformation. By this convention, we have to put an upper index on the coordinate differentials dx^i.

The coordinates themselves do not form a contravariant vector since the transformation between x^i and x'^i is in general nonlinear. Think of $x^i = (x\,y\,z)$ and $x'^i = (r\,\theta\,\phi)$. So we don't have to use an upper index on the coordinates x^i. But since dx^i <u>is</u> a contravariant vector, it makes sense to use an upper index also for x^i. Now another...

<u>Def</u>. A set of n qtys B_i, $i = 1, \ldots, n$, such that

$$B'_i = \frac{\partial x^j}{\partial x'^i} B_j$$

is said to form (or "transform as") a <u>covariant</u> <u>vector</u>.
It is customary to use ~~covariant~~ <u>lower</u> indices on a covariant vector.

For example, let f be a scalar, so $f(x) = f(x')$, and define

$$B_i = \frac{\partial f}{\partial x^i}, \qquad B'_i = \frac{\partial f}{\partial x'^i}.$$

Then by the chain rule,

$$B'_i = \frac{\partial f}{\partial x'^i} = \frac{\partial x^j}{\partial x'^i}\frac{\partial f}{\partial x^j} = \frac{\partial x^j}{\partial x'^i} B_j.$$

Calling the vector with components $\partial f/\partial x^i$ "the gradient of f" we see that the <u>gradient</u> <u>of</u> <u>a</u> <u>scalar</u> is a <u>covariant</u> <u>vector</u>.

In 3D space with coordinates $x^i = (x\,y\,z)$, the covariant components of ∇f are the same as the usual ones, since

$$(\nabla f)_i = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right), \qquad \text{and } \nabla f = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}.$$

But this is not true in coordinates $x^i = (r\,\theta\,\phi)$, since

$$(\nabla f)_i = \left(\frac{\partial f}{\partial r}, \frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial \phi}\right), \qquad \text{while}$$

means, covariant components.

$$\nabla f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\phi}.$$

Now let A^i be a contravariant vector and B_i be a covariant vector, and consider $A^i B_i$ (sum on i, naturally). Notice how this transforms:

$$A'^i B'_i = \frac{\partial x'^i}{\partial x^j} A^j \frac{\partial x^k}{\partial x'^i} B_k .$$

But

$$\frac{\partial x'^i}{\partial x^j} \frac{\partial x^k}{\partial x'^i} = \delta^k_j \qquad \text{(inverse Jacobians)}$$

so

$$A'^i B'_i = A^j B_k \delta^k_j = A^j B_j = A^i B_i .$$

The qty. $A^i B_i$ is a scalar. It is called the <u>contraction</u> of a contravariant vector with a covariant vector. Notice that it involves a summation over one contravariant and one covariant index.

$A^i B_i$ is a scalar because the contravariant index transforms by the Jacobian $\frac{\partial x'}{\partial x}$ while the covariant index transforms by the inverse Jacobian $\frac{\partial x}{\partial x'}$, and these factors cancel.

If A^i, B^i were <u>both</u> contravariant vectors and we formed the sum

$$A^i B^i ,$$

the result would <u>not</u> be equal to $A'^i B'^i$. Expressions like this may occur if we are working in a specific coordinate system, but they do not retain their form under changes of coordinates. For that we need to pair upper, lower indices when summing.

The sum $A^i B_i$ can be regarded as the "scalar product" of a contravariant with a covariant vector, but we have as yet no way of taking the scalar product of two contravariant or two covariant vectors.

Notice in the expression for the gradient as a covariant vector,

$$\underset{\substack{\uparrow \\ \text{lower on } B}}{B_i} = \frac{\partial f}{\partial x^i} \leftarrow \text{upper on } x$$

that an upper index on x in the denominator transforms as a lower index on the whole derivative. This is a simple rule: We switch the position of an index on going from a denominator to the whole fraction. Now, another def...

<u>Def</u>: A set of n^2 qtys C^{ij} such that

$$C'^{ij} = \frac{\partial x'^i}{\partial x^k}\frac{\partial x'^j}{\partial x^l} C^{kl}$$

is said to form a <u>second rank</u>, <u>contravariant</u> <u>tensor</u>.

Each contravariant index transforms as by the forward Jacobian matrix.

An example. Let A^i, B^i be two contravariant vectors, and define

$$C^{ij} = A^i B^j.$$

Such a tensor is said to be the <u>tensor product</u> or <u>outer</u> product of the two vectors. In general, the tensor product just means multiplying the components of two tensors of ranks r and s together, to get the components of a tensor of rank $r+s$.

Not every 2nd rank contravariant tensor can be written as the outer product of two contrav. vectors, but it can be written as a sum of such products.

This example (the outer product) of C^{ij} doesn't show very well what the use of such tensors is. We'll give a more interesting example (the metric tensor) in a moment.

<u>Def</u>. A set of n^2 qtys D_{ij} such that

$$D'_{ij} = \frac{\partial x^k}{\partial x'^i} \frac{\partial x^l}{\partial x'^j} D_{kl}$$

is said to form a <u>second rank, covariant tensor</u>. It transforms by one copy of the inverse Jacobian for each index.

<u>Def</u>: A set of n^2 qtys $E^{i}_{\cdot j}$ such that

$$E'^{i}_{\cdot j} = \frac{\partial x'^i}{\partial x^k} \frac{\partial x^l}{\partial x'^j} E^{k}_{\cdot l}$$

is said to form a <u>2nd rank, mixed tensor</u>. It transforms by one copy of the forward Jacobian for the contrav. index, and one of the inverse Jacobian for the covariant index.

If we ~~form~~ set $i=j$ on a mixed tensor $E^{i}_{\cdot j}$ to get $E^{i}_{\cdot i}$ then (by summation convention) the sum is implied, and we get the trace of E. This is a kind of contraction, and is a scalar, i.e. $E^{i}_{\cdot i} = E'^{i}_{\cdot i}$. But the trace of a purely contravariant or covariant tensor, C^{ii} or D_{ii}, is not an invariant (a scalar).

We use the dot in $E^{i}_{\cdot j}$ as a place holder, to indicate that the upper index is the first one. There is no reason however why the upper index must come first. For example, we could have a mixed tensor $F_{i\cdot}^{\cdot j}$ (2nd index upper).

An example of a mixed tensor is the Kronecker tensor. It has the value δ^i_j (usual 1s and 0s) in all coordinate systems. We don't need any dots to show which index comes first because it is symmetric in i, j. To show that these qtys, so defined, transform as a tensor, let

$$T^{i}_{\cdot j} = \delta^i_j$$

in coordinates x^i, and let's compute $T'^{i}_{\cdot j}$ in coords x'^i:

$$T'^{i}_{\cdot j} = \frac{\partial x'^i}{\partial x^k}\frac{\partial x^l}{\partial x'^j}\delta^k_l = \frac{\partial x'^i}{\partial x^k}\frac{\partial x^k}{\partial x'^j} = \delta^i_j .$$

On the other hand, if we try this with a covariant tensor, it doesn't work. That is, suppose

$$T_{ij} = \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases}$$

in coords x^i. Then you will find that in coords x'^i, T'_{ij} does not have the form of 1s and 0s (the same numerical components in all coordinates).

An equation

$$T_{ij} = \delta_{ij}$$

is <u>not covariant</u>, that is, it does not have the <u>same form</u> in all coordinates. It may be an expression of the numerical values of a tensor in one, specific coordinate system, but the equation does not retain its form in other coordinates.

Here the word "covariant" is being used in another sense.

A <u>covariant vector</u> = a vector with a certain <u>transformation law</u>.

A <u>covariant equation</u> has the <u>same form</u> in all coordinates.

It is desirable to express the laws of physics in covariant form, because that shows how the fundamental physics is independent of the coordinates we use to describe it.

Generalizing, a <u>tensor of rank r</u> has r indices, some contrav. and some covariant, with each contravariant index transforming by the forward Jacobian and each covariant index transforming by the inverse Jacobian. For example, $A_{ij\cdot}{}^{k}$ a 3rd rank tensor.

If we contract (sum) over pairs of upper, lower indices, we get a tensor in the remaining indices of lower rank. For example, if

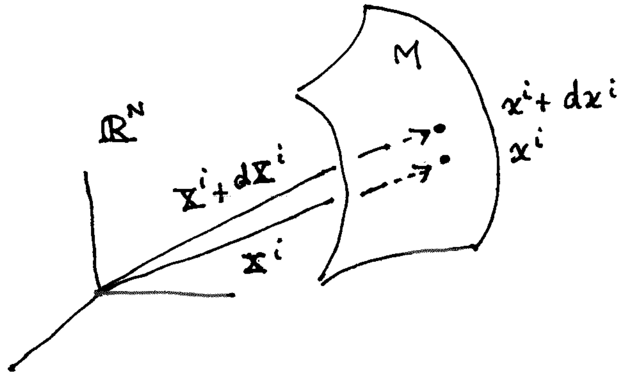
$$A^i = B^{ijk} \, C_{jk}$$

then A^i is a contrav. vector.

Now suppose our manifold M is a submanifold of N-dimensional Euclidean space $\mathbb{R}^N$, where

$$n = \dim M < N.$$

a picture:



Let X^i, $i = 1, \dots, N$ be the usual Euclidean coordinates on $\mathbb{R}^N$, so the dist^2 from the origin to a point is

$$\text{dist}^2 = \underbrace{\sum_i (X^i)^2}_{\text{no summ. convention}} = \underbrace{X^i X^i}_{\text{summ. convention}}.$$

Similarly, the dist^2 between two nearby pts with coordinates X^i and $X^i + dX^i$ is

$$ds^2 = dX^i dX^i$$

As for M, let x^i, $i = 1, \dots, n$ be coordinates on it. The equations of the surface are functions,

$$X^i = X^i(x), \qquad i = 1, \dots, N$$

that tell us where in $\mathbb{R}^N$ the point of M is that has coords x^i. For example, letting $X^i = (x, y, z)$ in 3D space, M = sphere of radius 1, $x^i = (\theta, \phi)$ = coords on sphere, we have

$$\left.\begin{aligned} x &= \sin\theta\cos\phi \\ y &= \sin\theta\sin\phi \\ z &= \cos\theta \end{aligned}\right\}.$$

Now if x^i and $x^i + dx^i$ are two nearby points on M, then the dist^2 between them is

$$ds^2 = dX^i dX^i = \frac{\partial X^i}{\partial x^k}\frac{\partial X^i}{\partial x^l}\, dx^k dx^l .$$

BTW, notice that $ds^2 = dX^i dX^i$ is not a covariant expression. That's all right, because we only intend to use this in one coordinate system, namely, Euclidean X^i on $\mathbb{R}^N$.

Now define

$$g_{kl} = \frac{\partial X^i}{\partial x^k}\frac{\partial X^i}{\partial x^l},$$

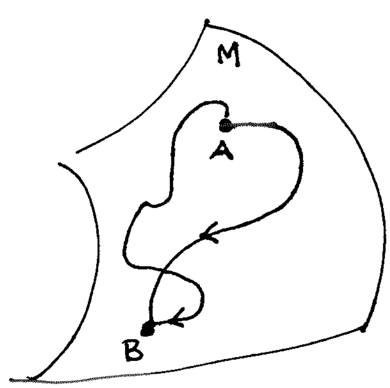
Note, $g_{kl} = g_{lk}$, g is <u>symmetric</u>

so that

$$\boxed{ds^2 = g_{kl}\, dx^k dx^l}$$

g_{kl} is called the <u>metric tensor</u> on M. It gives us the distance between any two nearby points on M in terms of the coordinates x^i. Notice that it does not give us the distance between two distant points of M, at least not directly; that is because the distance between distant points of M <u>depends on the path</u>.

(You can get the distance along a given path by integration.)



Note that g_{kl} depends on the point of M where we are, i.e.,

$$g_{kl} = g_{kl}(x)$$

It's easy to show that $g_{k\ell}$ really is a tensor. Let x^i and x'^i be two coords on M, so that

$$g_{k\ell} = \frac{\partial X^i}{\partial x^k}\frac{\partial X^i}{\partial x^\ell} = \frac{\partial X^i}{\partial x'^m}\frac{\partial X^i}{\partial x'^n}\frac{\partial x'^m}{\partial x^k}\frac{\partial x'^n}{\partial x^\ell}$$

$$= \frac{\partial x'^m}{\partial x^k}\frac{\partial x'^n}{\partial x^\ell}\, g'_{mn}$$

which is the transformation law for a 2nd rank, covariant tensor.

~~Next~~ We have derived the metric tensor for a manifold imbedded in a higher dimensional Euclidean space, but once we have $g_{k\ell}(x)$, it makes reference only to the coordinates x^i on M. The imbedding space and its coordinates do not appear. This leads to the idea that $g_{k\ell}(x)$ defines the intrinsic geometry of the surface.

Alternatively, we may imagine that we have a manifold M upon which a symmetric, covariant tensor $g_{ij}(x)$ is simply given, and we interpret $ds^2 = g_{ij}\,dx^i\,dx^j$ as the square distance between nearby points. (We must also require $\det g_{ij} \neq 0$.)

This is the usual point of view in relativity theory — we do not necessarily think of space-time as being imbedded in a higher dimensional space.

Now we define g^{ij} (with contravariant indices) as the inverse of g_{ij}, so that

$$g^{ik}\, g_{kj} = \delta^i_j .$$

As it turns out, g^{ij} defined this way transforms as a tensor, a contravariant, 2nd rank tensor. It is the <u>contravariant</u> metric tensor.

For example, in 3D space in Euclidean coordinates $x^i = (x\,y\,z)$, we have

$$ds^2 = dx^2 + dy^2 + dz^2, \quad \text{so} \quad g_{ij} = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix}, \text{ coords } (x\,y\,z).$$

It is the identity matrix, but if we write $g_{ij} = \delta_{ij}$ we have a non-covariant statement (not valid in all coordinate systems). For example, in spherical coordinates $(r\,\theta\,\phi) = x^i$, we have

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2, \quad \text{so}$$

$$g_{ij} = \begin{pmatrix} 1 & & 0 \\ & r^2 & \\ 0 & & r^2 \sin^2\theta \end{pmatrix} \text{ in coords } (r\,\theta\,\phi).$$

Then the inverse is easy to compute,

$$g^{ij} = \begin{pmatrix} 1 & & 0 \\ & 1/r^2 & \\ 0 & & \frac{1}{r^2 \sin^2\theta} \end{pmatrix}.$$

If A^i is a contravariant vector, then the qtys $g_{ij} A^j$ is a contraction on j so transforms as a covariant vector in i. It is customary to use the same symbol A for this qty, just with a covariant index. We write

$$A_i = g_{ij} A^j$$

and say that we have <u>lowered the index</u> on A. Equivalently, we have mapped a contravariant vector into an associated covariant one. For example, the covariant components of the velocity in spherical coordinates are

$$v_i = g_{ij} v^j = (\dot{r},\ r^2\dot{\theta},\ r^2\sin^2\theta\,\dot{\phi}).$$

Conversely, if B_i is a covariant vector, we write

$$B^i = g^{ij} B_j.$$

and say that we have <u>raised the index</u>, converting a covariant vector into a contravariant one. If you raise and then lower an index you return to where you started, since g^{ij} and g_{ij} are inverse matrices.

Similarly, you can raise any lower any index on any tensor, for example,

$$C_i{}^j = g_{ik} C^{kj}$$

$$C_{ij} = g_{jl}\ C_i{}^l = g_{ik}\, g_{jl}\ C^{kl}$$

etc.

If dx^i represent the coordinate difference between two nearby points, i.e., a small vector connecting those two points, then we can interpret

$$ds^2 = g_{ij}\ dx^i dx^j$$

as the <u>scalar product</u> of dx^i with itself. Notice this is the scalar product of <u>two</u> <u>contravariant</u> vectors with one another, something we couldn't do before we had a metric. More generally, if A^i, B^i are two contravariant vectors, we will define their scalar product as

$$(A,B) = A^i g_{ij} B^j = (B,A).$$

Notice that by the rules for lowering an index, this can also be written

$$(A,B) = A^i B_i$$

the invariant product of a contrav. and cov. vector that we had before. This can also be written as

$$(A,B) = g^{ik} A_k\, g_{ij}\, g^{jl} B_l = A_k\, \delta^k_j\, g^{jl} B_l$$

$$= A_j\, g^{jl} B_l = A_i\, g^{ij} B_j$$

which gives meaning to the scalar product of two covariant vectors.

Sometimes the contravariant and covariant transformation laws are more transparent in matrix form. If we arrange the components of A^i and B_j (a contrav. and cov. vector) as column vectors A and B, and the forward and inverse Jacobians as matrices

row $\downarrow$ row $\downarrow$

$$J^i{}_{\cdot j} = \frac{\partial x'^i}{\partial x^j}, \qquad (J^{-1})^i{}_{\cdot j} = \frac{\partial x^i}{\partial x'^j}$$

$\uparrow$ col. $\uparrow$ col.

then the transformation laws,

$$A'^i = \frac{\partial x'^i}{\partial x^j} A^j$$

$$B'_i = \frac{\partial x^j}{\partial x'^i} B_j$$

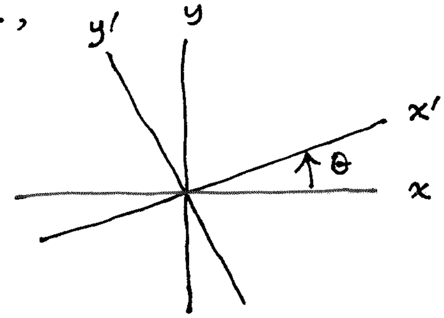
become

$$A' = JA, \qquad B' = J^{-1T} B.$$

(B transforms by the transpose of the inverse matrix). Then

$$(A,B) = (B,A) = B^T A = B'^T A' = B^T J^{-1} J A.$$

Now take an important special case, M = the Euclidean space $\mathbb{R}^n$ on which we use Euclidean coordinates x^i. Let us restrict ourselves to only Euclidean coordinates on this space. Such coordinates are related by orthogonal transformations, for example when $n=2$,



or more generally,

$$x'^i = R_{ij} x^j$$

where R is orthogonal, $R^T R = I$. In these coordinates, the metric is

$$ds^2 = dx^i dx^i = dx'^i dx'^i, \qquad \text{so}$$

$$g_{ij} = g'_{ij} = \delta_{ij}.$$

Earlier we said this was not a covariant eqn, and it's not, not under general (nonlinear) transformation. But it <u>is</u> covariant (has the same form) under linear, orthogonal transformation. What is covariant depends on the class of transformations we are considering.

Also, under these orthogonal transformations, the Jacobian matrix is

$$J^i{}_{\cdot j} = \frac{\partial x'^i}{\partial x^j} = R_{ij},$$

so

$$J^{-1T} = R^{-1T} = R.$$

Thus, contravariant and covariant vectors transform in the same way, and there is no point in distinguishing between them. So we might as well use uniformly lower indices, and declare, for example, that a vector A_i is something that transforms as

$$A'_i = R_{ij} A_j$$

under the coordinate transformation $x'_i = R_{ij} x_j$. In fact, we see that the coordinates themselves transform as a vector. Similarly, we can define tensors by how they transform. The metric $g_{ij} = \delta_{ij}$ is a tensor becase

$$g'_{ij} = R_{ik} R_{jl}\, g_{kl} = R_{ik} R_{jl}\, \delta_{kl} = R_{ik} R_{jk} = \delta_{ij}.$$

This is what everyone does when doing tensor analysis on Euclidean $\mathbb{R}^n$ in Euclidean (Cartesian, orthonormal) Coordinates. But if you use nonlinear or even nonorthogonal but linear coordinates on Euclidean $\mathbb{R}^n$, or any coordinates at all on a space that is not flat, then you may have to distinguish between contravariant and covariant vectors and indices.

Another important case is space-time in special relativity. Here the ~~space~~ manifold is $\mathbb{R}^4$ topologically speaking, but it is not Euclidean, i.e., the obvious Euclidean metric $t^2+x^2+y^2+z^2$ has no physical significance so we ignore it. Instead, the metric is $t^2-x^2-y^2-z^2$ and the associated metric tensor is $g_{\mu\nu}=\eta_{\mu\nu}$, where

$$\eta_{\mu\nu} = \begin{pmatrix} +1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}, \quad \text{coords } x^\mu = (t, x, y, z).$$

The eqn. $g_{\mu\nu}=\eta_{\mu\nu}$ is not covariant under arbitrary, nonlinear transformations in $\mathbb{R}^4$, but it is covariant under linear Lorentz transformations of the form

$$x'^\mu = \Lambda^\mu{}_\nu \, x^\nu,$$

where $\Lambda^\mu{}_\nu$ is one of the Lorentz transformation matrices we discussed earlier. The physical meaning of these coordinate systems is that they label events by using a rectangular array of meter sticks stationary w.r.t. one another and an array of synchronized clocks. Such a system constitutes a Lorentz frame, and two Lorentz frames moving with constant velocity with respect to one another specifies a Lorentz transformation.

Under a Lorentz transformation, the Jacobian matrix is

$$J^{\mu}{}_{\cdot\nu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} = \Lambda^{\mu}{}_{\cdot\nu}$$

so, for example, we define a contravariant vector A^{μ} in special relativity as one that transforms according to

$$A'^{\mu} = \Lambda^{\mu}{}_{\cdot\nu} A^{\nu} .$$

Likewise, a covariant vector B_{μ} transforms according to

$$B'_{\mu} = (\Lambda^{-1})^{\nu}{}_{\cdot\mu} B_{\nu}$$

As for tensors, let $g_{\mu\nu} = \eta_{\mu\nu}$ in one coordinate system.

Then

$$g'_{\mu\nu} = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} g_{\alpha\beta} = (\Lambda^{-1})^{\alpha}{}_{\cdot\mu} (\Lambda^{-1})^{\beta}{}_{\cdot\nu} \eta_{\alpha\beta} .$$

But a basic property of Lorentz transformations is

$$\Lambda^{\sigma}{}_{\cdot\alpha} \eta_{\sigma\tau} \Lambda^{\tau}{}_{\cdot\beta} = \eta_{\alpha\beta}$$

so

$$g'_{\mu\nu} = (\Lambda^{-1})^{\alpha}{}_{\cdot\mu} (\Lambda^{-1})^{\beta}{}_{\cdot\nu} \Lambda^{\sigma}{}_{\cdot\alpha} \Lambda^{\tau}{}_{\cdot\beta} \eta_{\sigma\tau}$$

$$= \delta^{\sigma}_{\mu} \delta^{\tau}_{\nu} \eta_{\sigma\tau} = \eta_{\mu\nu} .$$

The equation $g_{\mu\nu} = \eta_{\mu\nu}$ is covariant under Lorentz transformations.