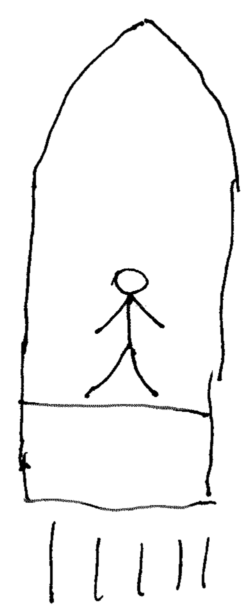
These notes only cover the part of the last lecture dealing with the uniformly accelerated, relativistic particle.

We are now talking about mechanics in special relativity. It is clear that Newtonian mechanics cannot be valid in SR, because if we apply a constant force to a particle, Newtonian mechanics would predict a constant acceleration, which would imply $v > c$ after a finite time. So how do we know what mechanical laws should replace $\vec{F} = m\vec{a}$ in SR? We use the fact that $\vec{F} = m\vec{a}$ is known to be correct at low velocities, more precisely, only when $\vec{v} = 0$, and then we use a Lorentz transfomation to find out what the acceleration is in a frame where $\vec{v} \neq 0$.

To imagine uniform acceleration in SR, think of a rocket ship whose engines are run at just the rate needed to make the passengers feel as if they are in Earth's gravity $g$:

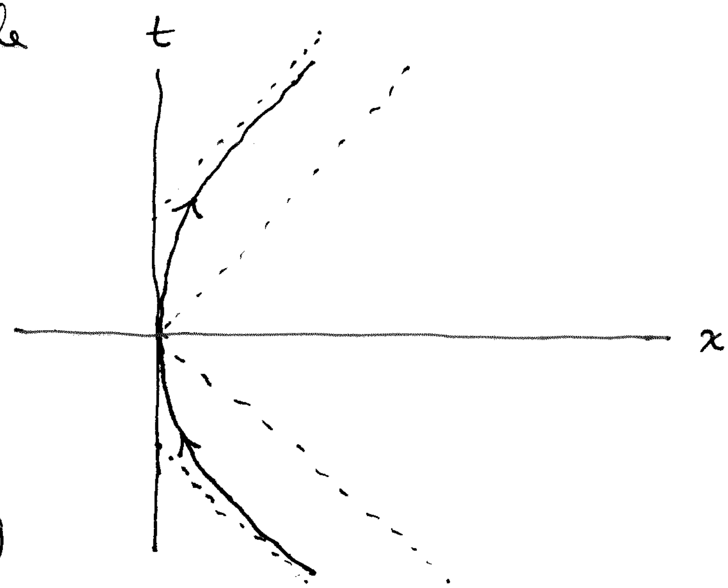
As the rocket uses up its fuel, its mass decreases, so we turn back the engines so that the passengers are comfortable in an artificial gravity of $g = 9.8\,m/sec^2$. The rocket has enough fuel so that it can reach relativistic velocities.



Another way to realize the same kind of motion is to put a charged particle in a uniform electric field. For example, if we charge a capacitor to $5 \times 10^5$ V and release an election from one plate, it will reach relativistic velocities before hitting the other plate.

Let's treat the uniform acceleration problem in 1D. Suppose $x=0$ at $t=0$ (and $v = \frac{dx}{dt} = 0$ @ $t=0$). As the particle accelerates, its world line bends over, approaching a $45^\circ$ slope relative to the time axis in a $1+1$ space time diagram.

The diagram makes it plausible that the world line of the particle will asymptote to a line parallel to the light cone emanating from $t=0$, $x=0$. Also shown is the world line of the decelerating particle for $t<0$, which asymptotes to another line parallel to the backward light cone when $t \to -\infty$.




Now to be general let $x^\mu(\tau)$ be the world line of a particle in any kind of motion. Recall that

$$d\tau^2 = dt^2 - dx^2,$$

so

$$\left(\frac{d\tau}{dt}\right)^2 = 1 - \left(\frac{dx}{dt}\right)^2 = 1 - v^2 = \frac{1}{\gamma^2},$$

where $v$ is the ordinary velocity $\frac{dx}{dt} = v$. Thus

$$\frac{dt}{d\tau} = \gamma.$$

Now in 1+1 dimensions $x^\mu = \begin{pmatrix} t \\ x \end{pmatrix}$, so the world velocity $u^\mu$ has components

$$u^\mu = \frac{dx^\mu}{d\tau} = \begin{pmatrix} \frac{dt}{d\tau} \\ \frac{dx}{d\tau} \end{pmatrix}.$$

But $\frac{dx}{d\tau} = \frac{dt}{d\tau}\frac{dx}{dt} = \gamma v$, so

$$u^\mu = \begin{pmatrix} \gamma \\ \gamma v \end{pmatrix}.$$

Recall that

$$u^\mu u_\mu = \frac{dx^\mu}{d\tau}\frac{dx_\mu}{d\tau} = \frac{d\tau^2}{d\tau^2} = 1.$$

Now define the <u>world</u> <u>acceleration</u> by

$$b^\mu = \frac{du^\mu}{d\tau} = \frac{dx^\mu}{d\tau^2}.$$

We use the symbol $b^\mu$ for the world acceleration, and reserve $a$ for the ordinary 3-acceleration $\frac{d^2x}{dt^2} = \frac{dv}{dt}$.

By differentiating $u^\mu u_\mu = 1$ w.r.t. $\tau$, we obtain

$$b^\mu u_\mu = 0.$$

The world acceleration is orthogonal to the world velocity. (Minkowski)

Next we have

$$b^\mu = \frac{du^\mu}{d\tau} = \frac{d}{d\tau}\begin{pmatrix} \gamma \\ \gamma v \end{pmatrix} = \begin{pmatrix} \frac{d\gamma}{d\tau} \\ \frac{d\gamma}{d\tau} v + \gamma \frac{dv}{d\tau} \end{pmatrix}.$$

Now $\frac{dv}{d\tau} = \frac{dt}{d\tau}\frac{dv}{dt} = \gamma a,$

and $\frac{d\gamma}{d\tau} = \frac{d}{d\tau}\left(\frac{1}{\sqrt{1-v^2}}\right) = \frac{v\,\frac{dv}{d\tau}}{(1-v^2)^{3/2}} = \gamma^4 v a.$

So,

$$b^\mu = \begin{pmatrix} \gamma^4 v a \\ \gamma^4 v^2 a + \gamma^2 a \end{pmatrix}.$$

Now notice that if $v = 0$, then $\gamma = 1$ and

$$b^\mu = \begin{pmatrix} 0 \\ a \end{pmatrix}.$$

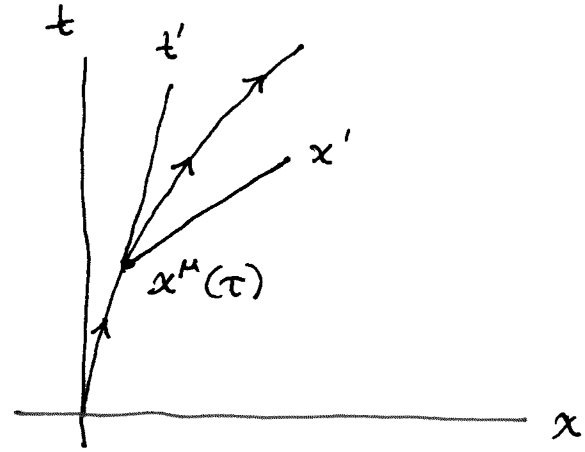
In this case, the spatial part of the world acceleration is the ordinary acceleration, but if $v \neq 0$ the relation is much more complicated.

Return now to the uniformly accelerated particle, and let $a =$ constant be the acceleration. This means that a passenger on the space ship ~~of~~ would feel an artificial gravitational force of $ma$.

Let $(x,t) = x^\mu$ be a fixed inertial frame and let $x'^\mu = (t', x')$ be the rest frame of the particle at some point on its world line.

At one instant, the particle is at rest in the rest frame, so

$$v' = \frac{dx'}{dt'} = 0,$$




and so Newton's laws are valid at that instant and

$$a' = \frac{d^2x'}{dt'^2} = a = \text{const.}$$

Thus

$$b'^\mu = \frac{dx'^\mu}{d\tau} = \begin{pmatrix} 0 \\ a \end{pmatrix} \quad \text{at one instant.}$$

But by Lorentz transforming back to frame $x^\mu$, we can get the world velocity in that frame:

$$b^\mu = \begin{pmatrix} \gamma & \gamma v \\ \gamma v & \gamma \end{pmatrix} \begin{pmatrix} 0 \\ a \end{pmatrix} = a \begin{pmatrix} \gamma v \\ \gamma \end{pmatrix}.$$

Also,

$$u^\mu = \begin{pmatrix} \gamma & \gamma v \\ \gamma v & \gamma \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma \\ \gamma v \end{pmatrix}$$

the latter of which we knew already. In applying these formulas, we have used the fact that $u^\mu$ and $b^\mu$ are 4-vectors (really 1+1 vectors), since $d\tau$ is a Minkowski scalar.

Now breaking these into components, we have

$$b^0 = \frac{d^2t}{d\tau^2} = a\gamma v = a\frac{dx}{d\tau} = a u^1$$

$$b^1 = \frac{d^2x}{d\tau^2} = a\gamma = a\frac{dt}{d\tau} = a u^0.$$

Clean this up:

$$\left.\begin{aligned} \frac{d^2t}{d\tau^2} &= a\frac{dx}{d\tau} \\ \frac{d^2x}{d\tau^2} &= a\frac{dt}{d\tau} \end{aligned}\right\}.$$

We wish to solve these subject to the initial conditions $x=0$, $\frac{dx}{d\tau}=0$ @ $\tau=0$, $\left(\text{also } t=0,\ \frac{dt}{d\tau}=1\right)$. Integrating once we find

$$\frac{dt}{d\tau} = ax + c_1,$$

$$\frac{dx}{d\tau} = at + c_2,$$

where $c_1, c_2$ are constants. Applying the initial conditions, we find $c_1 = 1$, $c_2 = 0$, so

$$\frac{dt}{d\tau} = ax + 1,$$

$$\frac{dx}{d\tau} = at.$$

To proceed it is easiest to plug the 2nd of these back into $\frac{d^2t}{d\tau^2} = a\frac{dx}{dt}$, giving

$$\frac{d^2t}{d\tau^2} = a^2 t,$$

or

$$t = A\cosh a\tau + B\sinh a\tau,$$

$$\frac{dt}{d\tau} = aA\sinh a\tau + aB\cosh a\tau,$$

where $A, B$ are new constants. Applying the initial conditions gives $A = 0$, $B = \frac{1}{a}$, so

$$\boxed{\begin{aligned} t &= \frac{1}{a}\sinh a\tau \\ \gamma = \frac{dt}{d\tau} &= \cosh a\tau \end{aligned}}$$

To get $x$ we substitute back, obtaining

$$\frac{dx}{d\tau} = at = \sinh a\tau,$$

or

$$\boxed{\begin{aligned} x &= \frac{1}{a}(\cosh a\tau - 1) \\ \frac{dx}{d\tau} &= \sinh a\tau \end{aligned}}$$