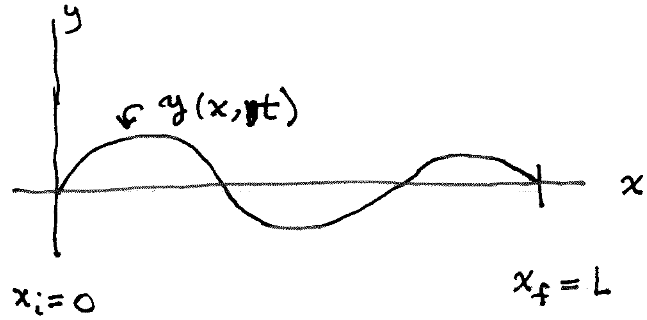
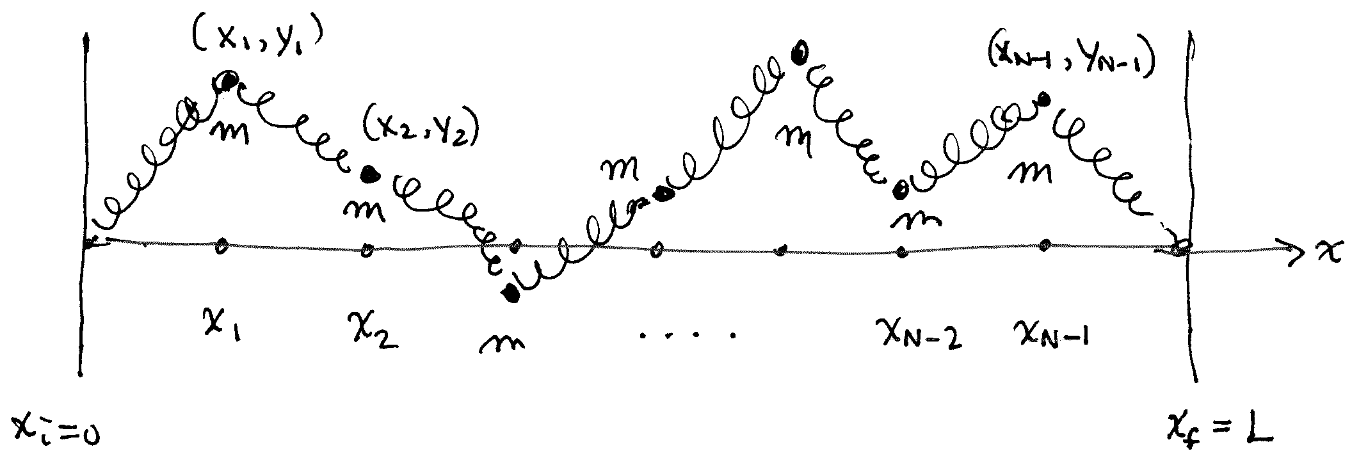
Now the vibrating string, an introduction to classical and quantum field theory.

An elastic string is stretched between $x=0=x_i$ and $x=L=x_f$ on the x-axis. When it is plucked, it vibrates in the y-direction. The position of the string at time t is given by a function



$y(x,t)$. We wish to find equations of motion for this function.

We model the string by a discrete set masses m uniformly distributed between $x_i=0$ and $x_f=L$. There are $N-1$ masses at positions $x_1, x_2, \ldots, x_{N-1}$, where $\Delta x = x_{i+1}-x_i$ is the spacing. $\Delta x = L/N$. We model the string by a set of identical, massless springs with spring constant k.



For simplicity we assume each mass is free to move only in the y-direction. This means that we are looking at only one transverse polarization or mode of oscillation. A real string could also move in the x- and z-directions. (The x-direction would be a longitudinal polarization.)

We find the equations of motion by using a Lagrangian. In this case, $L = T - V$. The kinetic energy is

$$T = \sum_{i=1}^{N-1} \frac{m}{2} \dot{y}_i^2$$

We only sum from $i=1$ to $N-1$ because there are no masses at $x_i = 0 = x_0$ and $x_f = L = x_N$. Only $\dot{y}_i$ appears (not $\dot{x}_i$) because we are constraining the motion to take place in the y-direction only.

The potential energy is

$$V = \sum_{i=0}^{N-1} \frac{k}{2}\left[(y_{i+1}-y_i)^2 + (x_{i+1}-x_i)^2 \right]$$

where we assume the potential energy of one spring is $\frac{k}{2} l^2$ where l is the length of the spring. We sum from $i=0$ to $N-1$ to cover all the springs. (The number of springs is one more than the number of masses m.).

The second term in the potential energy

$$\sum_{i=0}^{N-1} \frac{k}{2} (x_{i+1}-x_i)^2 = \sum_{i=0}^{N-1} \frac{k}{2} \Delta x^2$$

is a constant, and we can drop it without affecting the equations of motion. The overall Lagrangian becomes

$$\text{Lagr} = \sum_{i=1}^{N-1} \frac{m}{2}\,\dot{y}_i^{\,2} - \sum_{i=0}^{N-1} \frac{k}{2}\left(y_{i+1}-y_i\right)^2 .$$

We write Lagr for the Lagrangian to avoid confusion with the length L of the string.

Now we wish to take the limit $N \to \infty$ to get a model of a real (continuous) string. If we double N, we must cut Δx in half, and cut m in half, too, so that

$$\mu = \frac{m}{\Delta x}$$

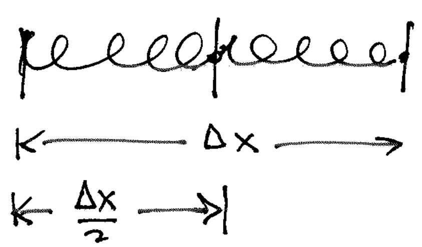
which is the linear mass density, should remain constant as $N \to \infty$. So we set $m = \mu\,\Delta x$, and

$$T = \sum_{i=1}^{N-1} \frac{m}{2}\,\dot{y}_i^{\,2} = \sum_{i=1}^{N-1} \Delta x\,\frac{\mu}{2}\,\dot{y}_i^{\,2}$$

$$\xrightarrow[(N\to\infty)]{} \int_0^L dx\,\frac{\mu}{2}\left(\frac{\partial y}{\partial t}\right)^2 .$$

The kinetic energy is a Riemann sum that goes to an integral in the limit $N \to \infty$.

Similarly, if we take a spring and cut it in half,



then it is intuitively clear that the spring constant k, which measures the stiffness of the spring, should double. Thus we expect k to be proportional to $1/\Delta x$, or

$$k = \frac{\kappa}{\Delta x}$$

where κ is a constant as $N \to \infty$. Thus the potential energy becomes

$$V = \sum_{i=0}^{N-1} \frac{k}{2} (y_{i+1} - y_i)^2 = \sum_{i=0}^{N-1} \frac{\kappa}{2\Delta x} (y_{i+1} - y_i)^2$$

$$= \sum_{i=0}^{N-1} \frac{\kappa}{2} \Delta x \left(\frac{y_{i+1} - y_i}{\Delta x}\right)^2 \xrightarrow[N\to\infty]{} \int_0^L dx \; \frac{\kappa}{2}\left(\frac{\partial y}{\partial x}\right)^2 .$$

Overall, the Lagrangian becomes

$$\text{Lagr} = \int_0^L dx \left[\frac{\mu}{2}\left(\frac{\partial y}{\partial t}\right)^2 - \frac{\kappa}{2}\left(\frac{\partial y}{\partial x}\right)^2\right]$$

$$= \int_0^L dx \;\; \mathcal{L}\left(\frac{\partial y}{\partial t}, \frac{\partial y}{\partial x}\right),$$

where

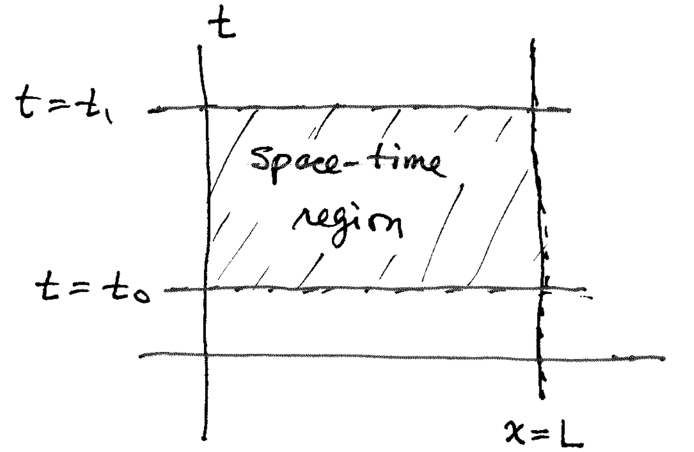
$$\mathcal{L} = \frac{\mu}{2}\left(\frac{\partial y}{\partial t}\right)^2 - \frac{\kappa}{2}\left(\frac{\partial y}{\partial x}\right)^2$$

is the <u>Lagrangian</u> <u>density</u>. Problems in discrete mechanics are expressed in terms of a Lagrangian, but in classical field theory we find a Lagrangian density whose spatial integral is the Lagrangian.

The <u>action</u> is the integral of Lagr. between two fixed times t_0 and t_1. It is a space-time integral of $\mathcal{L}$ over a space time region.

$$A[y(x,t)] = \int_{t_0}^{t_1} dt \int_0^L dx \quad \mathcal{L}\left(\frac{\partial y}{\partial t}, \frac{\partial y}{\partial x}\right).$$

In this case, the space-time region is a rectangle. The action is a functional of the function $y(x,t)$, a "history" of the string between t_0 and t_1. We can compute the action for any history, not only the physical ones. But according to Hamilton's principle, the histories that are solutions of the equations of motion are those that cause the action to be stationary, that is,



a small variation $\delta y(x,t)$ about a physical history $y(x,t)$ causes only second order variations in the action.

In discrete mechanics, the action $A[q(t)]$ is a functional of a "path" $q(t)$ in configuration space. The path is assumed to lie in a path space such that $q(t_0) = q_0$ and $q(t_1) = q_1$ are fixed endpoints. This is how Hamilton's principle works. Then the variations $\delta q(t)$ must satisfy $\delta q(t_0) = \delta q(t_1)$ so that both $q(t)$ and $q(t) + \delta q(t)$ can lie in the given path space.

Similarly, for the string, we assume that the histories $y(x,t)$ lie in some "history space" of functions such that

$y(x,t_0) = y_0(x)$ and $y(x,t_1) = y_1(x)$, for given initial and final conditions $y_0(x)$ and $y_1(x)$. This means that $\delta y(x,t_0) = \delta y(x,t_1) = 0$.

Also, the history space for our vibrating string satisfies $y(0,t) = y(L,t) = 0$, since the string is tied at its ends $x=0$ and $x=L$ (so y must be 0 there). This implies $\delta y(0,t) = \delta y(L,t) = 0$.

Altogether, δy vanishes at the boundary of the space-time region over which we integrate to get the action. The equations of motion are given by

$$\frac{\delta A}{\delta y(x,t)} = 0.$$

Let's generalize the situation and change notation before doing the functional derivative. Let's replace $y \to \psi$, where ψ stands for a generic field, and replace x by $\vec{x}$ (3D) to cover the case of fields $\psi(\vec{x},t)$ in 3D space. Let's also allow $\mathcal{L}$ to depend on ψ itself in addition to its space-time derivatives (recall we had $\mathcal{L}\left(\frac{\partial y}{\partial t}, \frac{\partial y}{\partial x}\right)$ for the string). Thus we have $\mathcal{L}\left(\psi, \frac{\partial \psi}{\partial t}, \nabla\psi\right)$, and

$$\text{Lagr} = \int d^3\vec{x}\ \mathcal{L}\left(\psi, \frac{\partial \psi}{\partial t}, \nabla\psi\right)$$

$$\text{Action} = A[\psi(\vec{x},t)] = \int dt \int d^3\vec{x}\ \mathcal{L} = \int d^4x\ \mathcal{L}.$$

The action is the integral of $\mathcal{L}$ over some space-time region. We assume that $\delta\psi(\vec{x},t)$ vanishes on the boundary of this region.

Now compute the functional derivative. We want

$$\delta A = A[\psi(\vec{x},t)+\delta\psi(\vec{x},t)] - A[\psi(\vec{x},t)]$$

$$= \int d^4x\, \mathcal{L}\left(\psi+\delta\psi, \frac{\partial\psi}{\partial t}+\frac{\partial}{\partial t}\delta\psi, \nabla\psi+\nabla\delta\psi\right) - \int d^4x\, \mathcal{L}\left(\psi,\frac{\partial\psi}{\partial t},\nabla\psi\right)$$

$$= \int d^4x \left[\frac{\partial\mathcal{L}}{\partial\psi}\delta\psi + \frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\psi}{\partial t}\right)}\frac{\partial}{\partial t}\delta\psi + \frac{\partial\mathcal{L}}{\partial(\nabla\psi)}\cdot\nabla\delta\psi\right] \qquad (*)$$

through first order in $\delta\psi$. The 3rd term means,

$$\frac{\partial\mathcal{L}}{\partial(\nabla\psi)}\cdot\nabla\delta\psi = \sum_{i=1}^{3}\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\psi}{\partial x_i}\right)}\frac{\partial}{\partial x_i}(\delta\psi).$$

Notice that $\mathcal{L}$ is an ordinary function of its arguments, ψ, $\frac{\partial\psi}{\partial t}$ and $\nabla\psi$, while the action is a functional of $\psi(\vec{x},t)$.

To compute the functional derivative, we must make the right hand side look like

$$\int d^4x\, \frac{\delta A}{\delta\psi(\vec{x},t)}\,\delta\psi(\vec{x},t).$$

We do this by integrating by parts, just like in discrete particle mechanics except now we have a spatial integral as well as a time integral. For the middle term we have

$$\int_{t_0}^{t_1} dt \, \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \psi}{\partial t}\right)} \frac{\partial}{\partial t}\delta\psi = \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \psi}{\partial t}\right)} \delta\psi \Bigg|_{t_0}^{t_1}$$

$$- \int_{t_0}^{t_1} dt \, \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \psi}{\partial t}\right)}\right) \delta\psi$$

Here we concentrate on just the time integral (ignoring $\int d^3\vec{x}$) and we assume that the space-time region is bounded by "planes" $t = t_0$ and $t = t_1$ (really hyperplanes in space-time). Then because $\delta\psi = 0$ at t_0 and t_1, the first term (boundary term) above vanishes, and only the integral remains.

As for the 3rd term in Eq. (*) (last page) it is

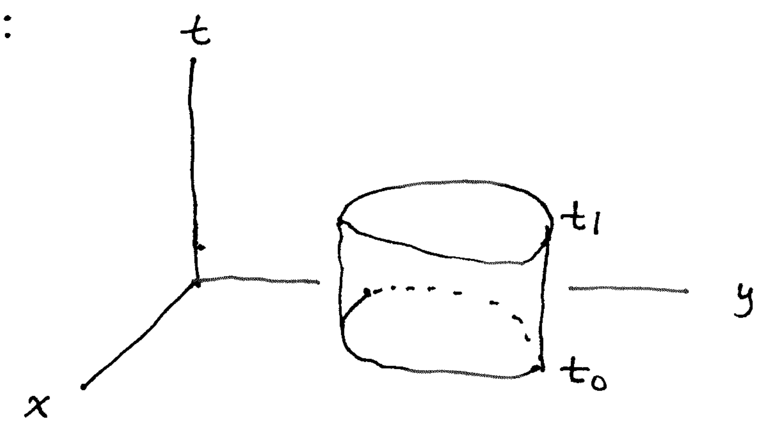
$$\int d^3\vec{x} \, \frac{\partial \mathcal{L}}{\partial(\nabla\psi)} \cdot \nabla\delta\psi = \int d^3\vec{x} \quad \nabla f \cdot \vec{A}$$

where

$$f = \delta\psi, \qquad \vec{A} = \frac{\partial \mathcal{L}}{\partial(\nabla\psi)}.$$

Also, we suppress the time integral, and just look at the spatial part of the integration. We assume the boundary of the spatial part of the space-time region is independent of time (the region is a kind of "cylinder":

(This applies in the case of the vibrating string, where the spatial part of the space-time



region is the fixed interval $[0,L]$ on the x-axis). Now use the identity,

$$\nabla\cdot(f\vec{A}) = \nabla f\cdot\vec{A} + f\,\nabla\cdot\vec{A},$$

so that

$$\int d^3\vec{x}\;\nabla f\cdot\vec{A} = \underbrace{\int d^3x\;\nabla\cdot(f\vec{A})}_{=\int_{surf} f\vec{A}\cdot d\vec{S}} - \int d^3\vec{x}\; f\,(\nabla\cdot\vec{A})$$

using Gauss' theorem in the first term. $d\vec{S}$ is the area element on the surface. Thus we have

$$\int d^3\vec{x}\;\frac{\partial\mathcal{L}}{\partial(\nabla\psi)}\cdot\nabla\delta\psi = \int_{surf}\delta\psi\;\frac{\partial\mathcal{L}}{\partial(\nabla\psi)}\cdot d\vec{S}$$

$$-\int d^3\vec{x}\;\nabla\cdot\left(\frac{\partial\mathcal{L}}{\partial(\nabla\psi)}\right)\delta\psi\,.$$

But assuming $\delta\psi = 0$ on the spatial part of the boundary, the surface integral vanishes and we have only the last integral. Altogether, we have

$$\delta A = \int d^4x\left[\frac{\partial\mathcal{L}}{\partial\psi} - \frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial(\frac{\partial\psi}{\partial t})}\right) - \nabla\cdot\left(\frac{\partial\mathcal{L}}{\partial(\nabla\psi)}\right)\right]\delta\psi,$$

so the functional derivative is

$$\frac{\delta A}{\delta \psi(\vec{x},t)} = \frac{\partial \mathcal{L}}{\partial \psi} - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \psi}{\partial t}\right)}\right) - \nabla\cdot\left(\frac{\partial \mathcal{L}}{\partial(\nabla\psi)}\right).$$

This must vanish on the physical motions, so the equations of motion must be

$$\boxed{\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \psi}{\partial t}\right)}\right) + \nabla\cdot\left(\frac{\partial \mathcal{L}}{\partial(\nabla\psi)}\right) = \frac{\partial \mathcal{L}}{\partial \psi}}$$

These are the Euler-Lagrange equations for a scalar field in 3D.

In the case of the vibrating string, the field $y(x,t)$ vanished at the spatial boundaries $x=0$ and $x=L$ because the string was pinned down there. For fields ψ in 3D, we often argue as follows. We say we are only interested in field configurations that go to zero at spatial infinity. Then we take the boundary of the spatial part of the space-time region to be a surface that goes $\to\infty$.

If the 3D field has multiple components ψ_i, $i=1,\dots,n$ (a vector or tensor field of some kind), then we just replace ψ in the E-L eqns above by ψ_i (and we get n E-L eqns). In this course we will stick to scalar fields as much as possible.

Let's revert to the vibrating string, for which the Euler-Lagrange equations are

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial y}{\partial t}\right)}\right) + \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial y}{\partial x}\right)}\right) = \frac{\partial \mathcal{L}}{\partial y},$$

where

$$\mathcal{L} = \frac{\mu}{2}\left(\frac{\partial y}{\partial t}\right)^2 - \frac{\kappa}{2}\left(\frac{\partial y}{\partial x}\right)^2 .$$

So,

$$\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial y}{\partial t}\right)} = \mu \frac{\partial y}{\partial t}, \quad \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial y}{\partial x}\right)} = -\kappa \frac{\partial y}{\partial x}, \quad \frac{\partial \mathcal{L}}{\partial y} = 0,$$

and the E-L eqns are

$$\mu \frac{\partial^2 y}{\partial t^2} - \kappa \frac{\partial^2 y}{\partial x^2} = 0,$$

or

$$\boxed{\frac{\partial^2 y}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} = 0} \qquad \text{Vibrating string.}$$

where $v^2 = \frac{\kappa}{\mu}$. The eqn for $y(x,t)$ is the <u>wave equation</u>, with phase velocity $v = \sqrt{\frac{\kappa}{\mu}}$. Notice that stiffer and lighter strings have higher phase velocities. If we substitute a solution $y(x,t) = e^{i(kx-\omega t)}$, we find $-k^2 + \frac{\omega^2}{v^2} = 0$ or $\boxed{\omega = vk}$. Such a solution does not satisfy the boundary conditions at $x=0$, $x=L$ and it is complex, not real, but it does give the right dispersion relation. The dispersion relation is the same as for light waves, $\omega = ck$, except c is replaced by v. (Of course, for real strings, $v \ll c$.)

It is convenient to do a normal mode expansion for the string. Because of the boundary conditions at $x=0$, $x=L$, $y(x,t)$ can be expanded in a sine Fourier series with time-dependent coefficients,

$$y(x,t) = \sqrt{\frac{2}{L}} \sum_{m=1}^{\infty} q_m(t) \sin\left(\frac{m\pi x}{L}\right)$$

where we split off a factor of $\sqrt{\frac{2}{L}}$ for later convenience. Here m stands for "mode", it is the mode index or mode number.

We substitute this Fourier series into the Lagrangian. In effect, we are doing a "change of coordinates" on configuration space, where we represent the wave by the infinite sequence of coefficients $(q_1, q_2, \ldots)$ instead of the function $y(x,t)$. Notice that the coefficients q_m are functionals of $y(x,t)$, since

$$q_m = \sqrt{\frac{2}{L}} \int_0^L dx \; y(x) \sin \frac{m\pi x}{L}.$$

This follows from the orthonormality relations,

$$\frac{2}{L} \int_0^L dx \, \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{m'\pi x}{L}\right) = \delta_{mm'}.$$

The mathematics is the same as for a particle in a box in quantum mechanics.

Differentiating, we have

$$\frac{\partial y}{\partial t} = \sqrt{\frac{2}{L}} \sum_{m=1}^{\infty} \dot{q}_m \sin\left(\frac{m\pi x}{L}\right),$$

so the kinetic energy is

$$T = \frac{\mu}{2}\int_0^L dx \left(\frac{\partial y}{\partial t}\right)^2 = \frac{\mu}{2}\cdot\frac{2}{L}\int_0^L dx \sum_{mm'} \dot{q}_m \dot{q}_{m'} \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{m'\pi x}{L}\right)$$

$$= \frac{\mu}{2}\sum_{mm'} \dot{q}_m \dot{q}_{m'}\, \delta_{mm'} = \frac{\mu}{2}\sum_{m=1}^{\infty} \dot{q}_m^2 .$$

As for the kinetic energy, we have

$$\frac{\partial y}{\partial x} = \sqrt{\frac{2}{L}} \sum_{m=1}^{\infty} q_m k_m \cos\left(\frac{m\pi x}{L}\right),$$

where we write $k_m = \frac{m\pi}{L}$ for the wave number of mode m. Then

$$V = \frac{\kappa}{2}\int_0^L dx \left(\frac{\partial y}{\partial x}\right)^2 = \frac{\kappa}{2}\,\frac{2}{L}\int_0^L dx \sum_{mm'} q_m q_{m'} k_m k_{m'} \cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{m'\pi x}{L}\right).$$

Now use

$$\frac{2}{L}\int_0^L dx \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{m'\pi x}{L}\right) = \delta_{mm'}$$

(just like the sine integral), so

$$V = \frac{\kappa}{2} \sum_{mm'} q_m q_{m'} k_m k_{m'} \; \delta_{mm'}$$

$$= \frac{\kappa}{2} \sum_{m=1}^{\infty} k_m^2 q_m^2 .$$

Overall, the Lagrangian is

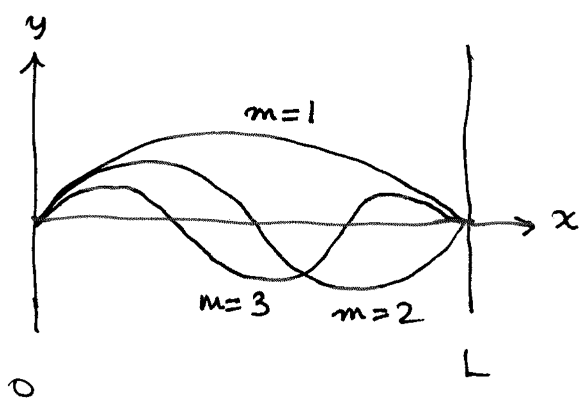
$$\text{Lagr} = \sum_{m=1}^{\infty} \left(\frac{\mu}{2} \dot{q}_m^2 - \frac{\kappa}{2} k_m^2 q_m^2 \right)$$

$$= \sum_{m=1}^{\infty} \left(\frac{\mu}{2} \dot{q}_m^2 - \frac{\mu}{2} v^2 k_m^2 q_m^2 \right)$$

since $v^2 = \frac{\kappa}{\mu}$

$$= \sum_{m=1}^{\infty} \left(\frac{\mu}{2} \dot{q}_m^2 - \frac{\mu \omega_m^2}{2} q_m^2 \right)$$

where $\omega_m = v k_m$ is the frequency of mode m with wave number $k_m = \frac{m\pi}{L}$. The Lagrangian is a sum of independent harmonic oscillators, one for each mode.



The modes.

To convert to the classical Hamiltonian, we define the momentum for mode m by

$$p_m = \frac{\partial \, Lagr}{\partial \dot{q}_m} = \mu \dot{q}_m ,$$

and

$$H = \sum_m p_m \dot{q}_m - L$$

$$= \sum_{m=1}^{\infty} \left(\frac{p_m^2}{2\mu} + \frac{\mu \omega_m^2}{2} q_m^2 \right),$$

a sum of an infinite number of harmonic oscillators, one for each mode. The oscillators are decoupled from each other.

We now quantize this Hamiltonian, using Dirac's prescription, which is to replace the classical q's and p's by operators,

$$q_m \longrightarrow \text{multiplication by } q_m$$

$$p_m \longrightarrow -i\hbar \frac{\partial}{\partial q_m}$$

so that

$$[q_m, p_{m'}] = i\hbar \, \delta_{mm'} .$$

These operators act on a space of wave functions $\Psi(q_1, q_2, \dots)$ that depend on all the q's. Equivalently, since the q_m's are functionals of $y(x,t)$, Ψ can be thought of as a functional of $y(x,t)$, $\Psi[y(x,t)]$. In quantum

~~the~~ field theory, the wave function is a functional of the classical field.

The energy eigenstates are just harmonic oscillator eigenstates, one for each mode. Each mode m has its own quantum number $n_m = 0, 1, 2, \ldots$, and contributes an energy $(n_m + 1/2)\hbar\omega_m$. The ~~to~~ energy in a given mode is quantized. The total energy is

$$E = \sum_{m=1}^{\infty} (n_m + 1/2)\hbar\omega_m .$$

In particular, the energy of the ground state is

$$E_{gnd} = \sum_{m=1}^{\infty} \tfrac{1}{2}\hbar\omega_m .$$

It is the sum of the zero-point energies, one for each mode.

Unfortunately, this energy diverges,

$$E_{gnd} = \sum_{m=1}^{\infty} \tfrac{1}{2}\hbar v k_m = \sum_{m=1}^{\infty} \left(\tfrac{1}{2}\hbar v \tfrac{\pi}{L}\right) m = \text{const} \sum_{m=1}^{\infty} m = \infty .$$

The divergence comes from the ∞ number of modes with large mode number (hence small wave length). In a real vibrating string (or elastic solid, which is more realistic) there is a cut-off on the mode number, since when k gets large enough that $\lambda = \frac{2\pi}{k}$ is small enough that it is comparable to the lattice spacing of the solid, then the continuum model for the vibrations breaks down.

But a similar sum occurs for quantum fields such as the electromagnetic field, where there is no underlying lattice. What limits the sum in that case? Further discussion will be given in class, and may be found in Ch. 1 of the book.