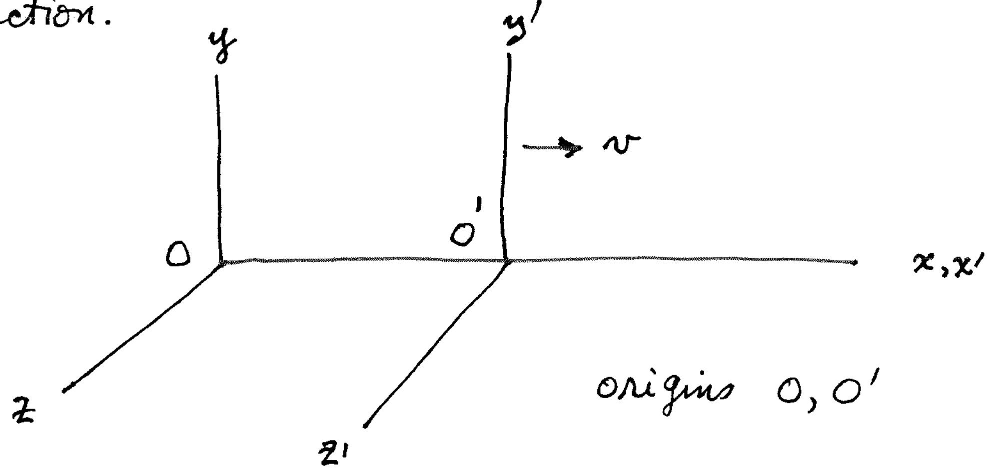
Consider 2 frames, xyz or unprimed or "stationary" frame, and x'y'z' or primed or "moving" frame in three-dimensional space. The primed frame is moving with velocity v down the x-direction.



In the stationary frame we fill space with an rectangular array of meter sticks for measuring coordinates. The meter sticks are stationary in that frame. We also fill up space with an array of clocks that are synchronized with one another and stationary in that frame. Similarly, we construct another array of meter sticks and clocks that are stationary in the moving frame. These clocks are synchronized in the moving frame. Somehow we keep the stationary meter sticks and clocks from colliding with the moving ones. When a moving clock passes a stationary clock, we can read off the times t, t' on the 2 clocks, as well as the coordinates xyz and $x'y'z'$ from the meter sticks.

According to the the special theory of relativity, the two sets of data $(xyzt)$ and $(x'y'z', t')$ are related by

This assumes $O = O'$ when $t = t' = 0$.

<u>A Lorentz transformation</u>

$$\left\{ \begin{aligned} x' &= \gamma(x - vt) \\ t' &= \gamma\left(t - \frac{vx}{c^2}\right) \\ y' &= y \\ z' &= z \end{aligned} \right.$$

where $\gamma = \dfrac{1}{\sqrt{1 - v^2/c^2}}$.

In the following we will choose units so that $c=1$. Since the velocity of the moving frame must be less than the velocity of light, we have $-1 < v < +1$ (v is the x-component of the velocity, a signed quantity). This simplifies the Lorentz transformation,

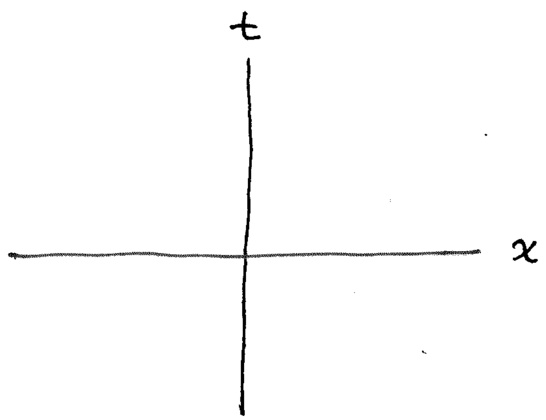
$$\begin{aligned} t' &= \gamma(t - vx) \\ x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \end{aligned}$$

The coordinates y, z perpendicular to the motion are not affected by the Lorentz transformation.

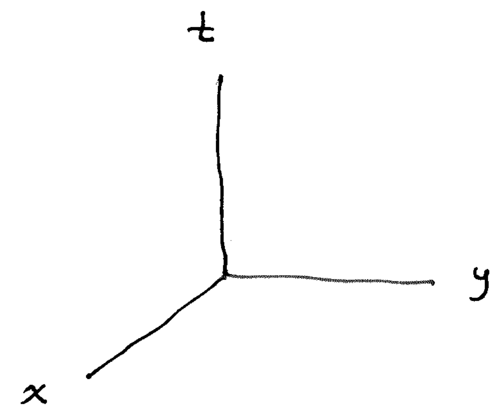
The most notable thing about the L.T. is the transformation of time. In pre-relativistic physics, it never occurred to anyone that time was not a universal quantity, the same for all observers. That is, people would have written just $t'=t$.

The coincidence of a stationary and a moving clock is an example of an <u>event</u>, something that happens at a particular place at a particular time. To specify an event requires 4 coordinates, $(xyzt)$ or $(x'y'z't')$, either one, since these are functions of each other under the L.T. Thus, the space of all events is 4-dimensional. This space is called <u>space-time</u>, and the Lorentz transformation can be regarded as a coordinate transformation on it.

Since space-time has 4 dimensions, it is difficult to make a drawing of it. In practice we usually suppress one or more of the spatial dimensions to make a drawing possible. If we suppress the y and z dimensions, we get a 1+1 dimensional version of space-time, if we suppress just z we get a 2+1 version:



1+1

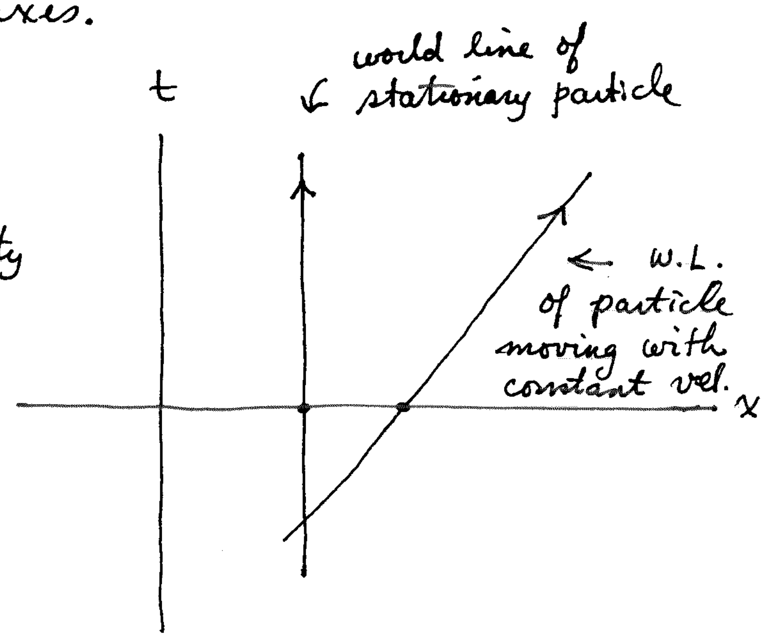


2+1

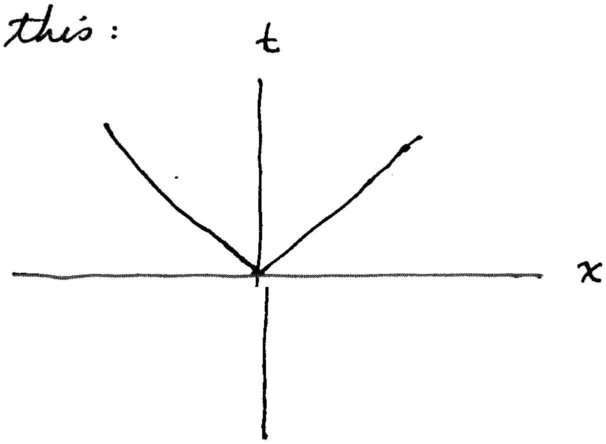
(Can't draw 3+1 version).

Work with 1+1 version. A particle that is stationary in the unprimed frame traces out a line in space-time that is parallel to the time axis.

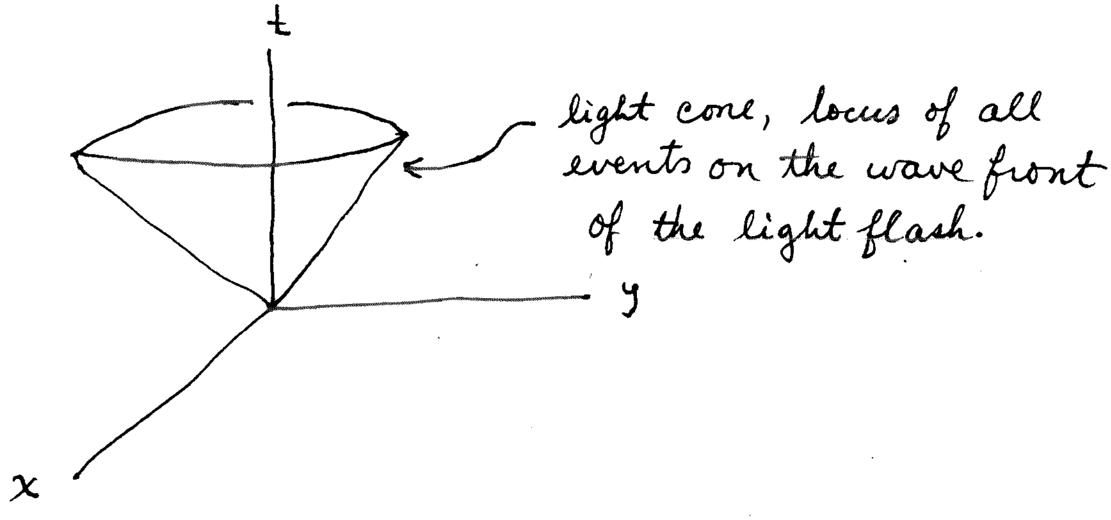
This line is the <u>world-line</u> of the particle. A particle that is moving at a constant velocity also has a straight world line, but it is sloped relative to the t-axis.



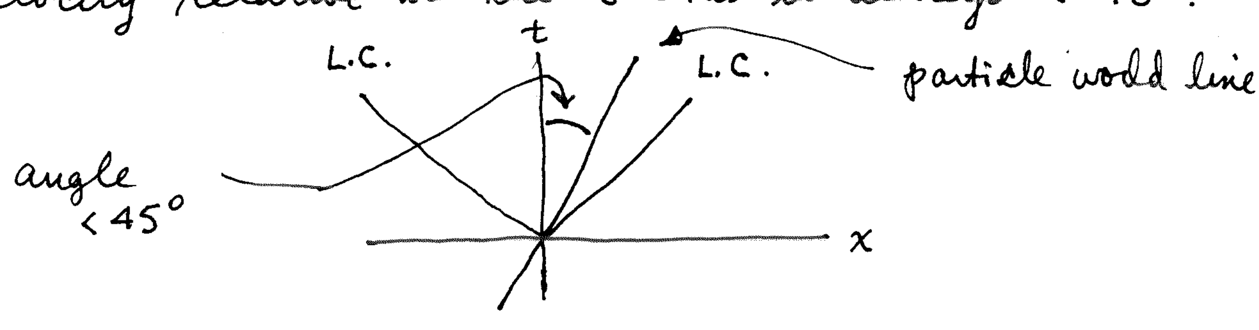
Suppose a flash of light is emitted at O at $t=0$. The wave front of the flash travels down the x-axis in both directions, so $x = \pm ct$ is the eqn of the wave front (just $x = \pm t$ in our units). So in a 1+1 space-time diagram it looks like this:



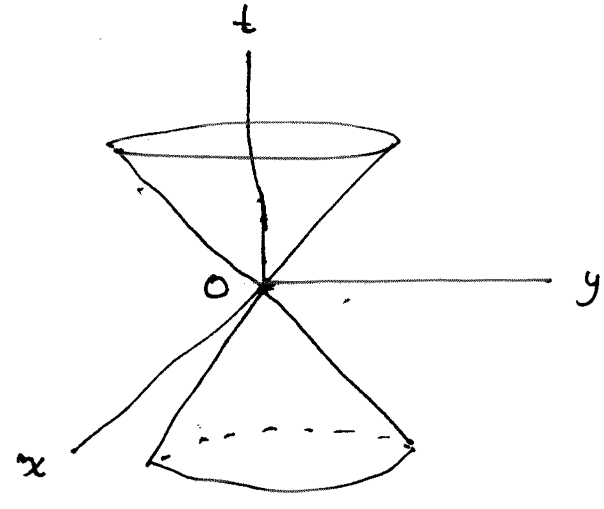
The 2 lines at 45° ∨ from the t-axis are the <u>light cone</u>. The terminology is more obvious in 2+1 dimensions,



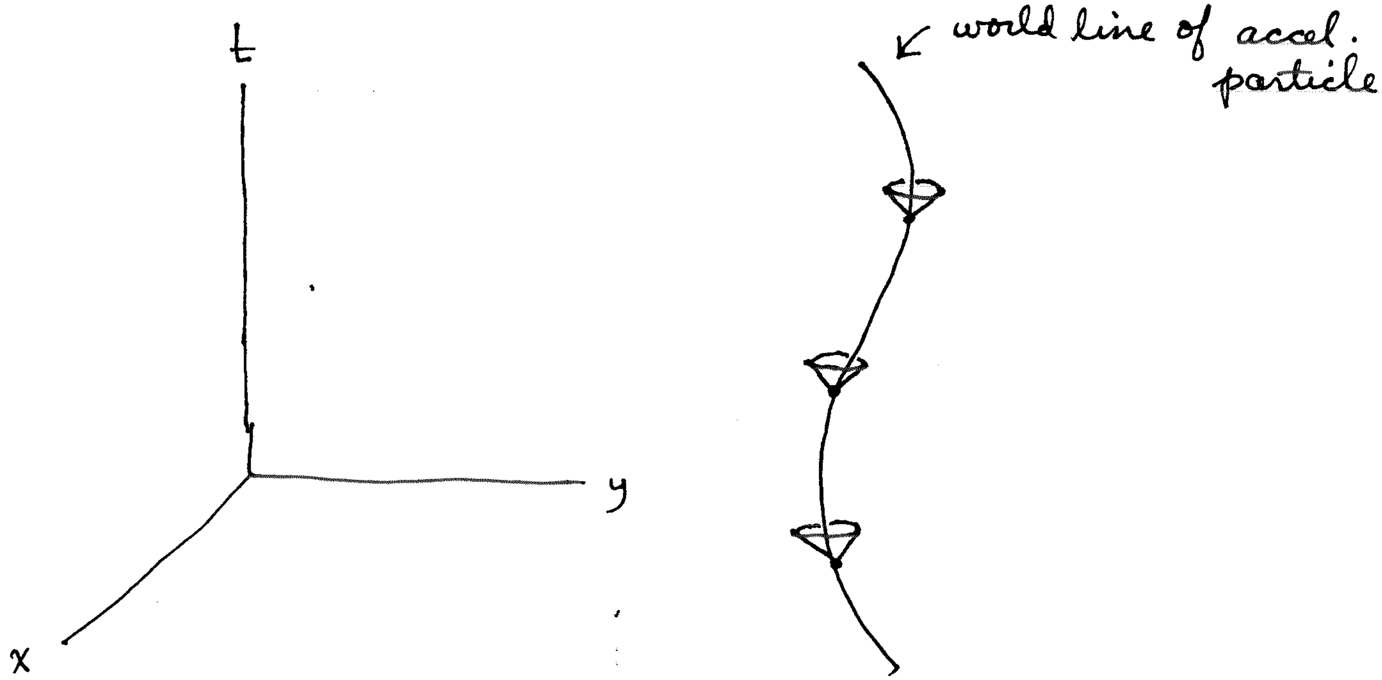
Since a particle can never have a velocity $\geq c$, the slope of a world line of a particle moving with constant velocity relative to the t-axis is always $< 45°$.



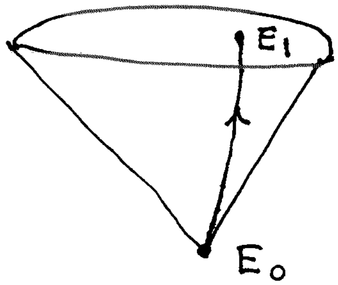
One can also speak of the <u>backward</u> light cone, produced by a spherical light wave front at $t < 0$ converging on the origin O.



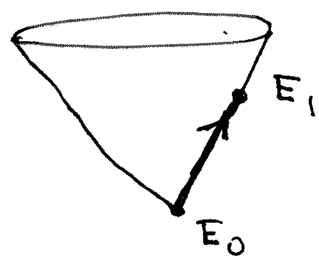
An accelerated particle has a curved world line. However, since the velocity is always $< c$, the slope of the curved world line is always $< 45°$ relative to the t-axis. Equivalently, if the particle is imagined to emit flashes of light at regular intervals, thereby creating a sequence of light cones along its world line, then the world line always moves into the <u>interior</u> of these (forward) light cones.



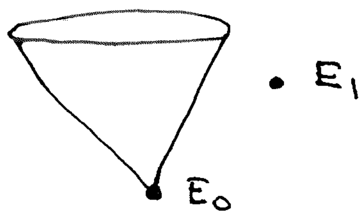
The interior of the forward light cone of a given event is the set of events that can be influenced by the given event.



For example, at event E_0 we can send a package traveling at velocity $v < c$ that can reach event E_1 and have an effect there — perhaps deliver some information or cause something to happen. Or, if event E_1 lies <u>on</u> the light cone, it can be reached from E_0 by a light signal.

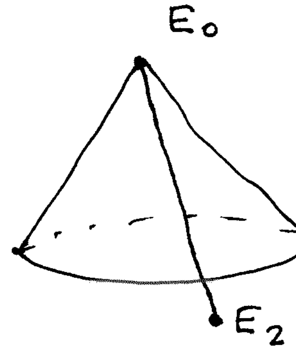


But if E_1 lies outside the light cone, then events E_0 and E_1 are <u>causally disconnected</u>. No decision made at E_0 can influence anything at E_1, because to get there would require moving with velocity $v > c$.



The forward light cone and its interior is the set of events that are <u>causally connected</u> with E_0.

Similarly, the backward light cone of E_0 is the set of events that can have an effect at E_0:

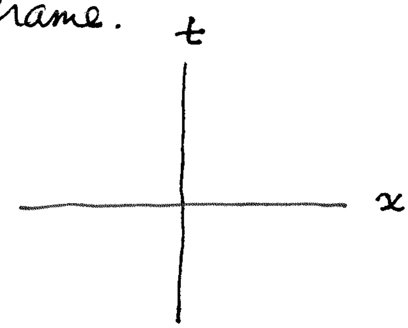


(E_2, E_0 connected by a particle trajectory with $v<c$.)

So far we have only plotted the unprimed frame in a space-time diagram. Now let's also plot the primed (moving) frame. Recall,

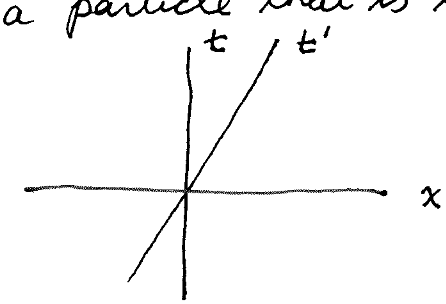
$$t' = \gamma(t - vx) \qquad \gamma = \frac{1}{\sqrt{1-v^2}}$$
$$x' = \gamma(x - vt)$$

The t-axis is the set of events for which $x=0$. It is the world line of a particle stationary at the origin of the unprimed frame.



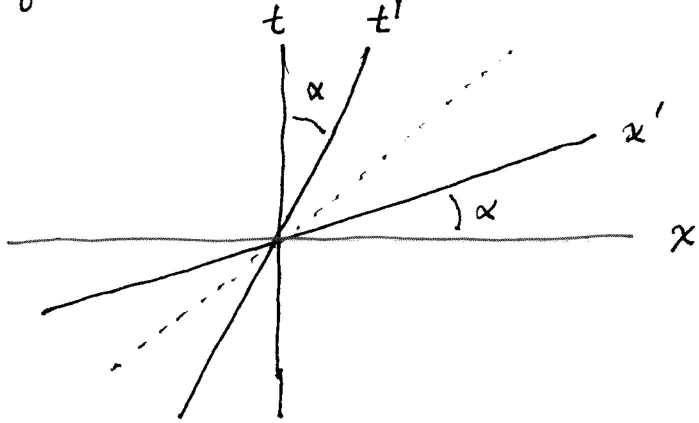
Similarly, the t' axis is the set of events for which $x'=0$. It is the world line of a particle that is stationary in the primed frame.

Setting $x'=0$, the L.T. gives $x = vt$.



$x = vt$ is the eqn. of the t'-axis in the unprimed frame.

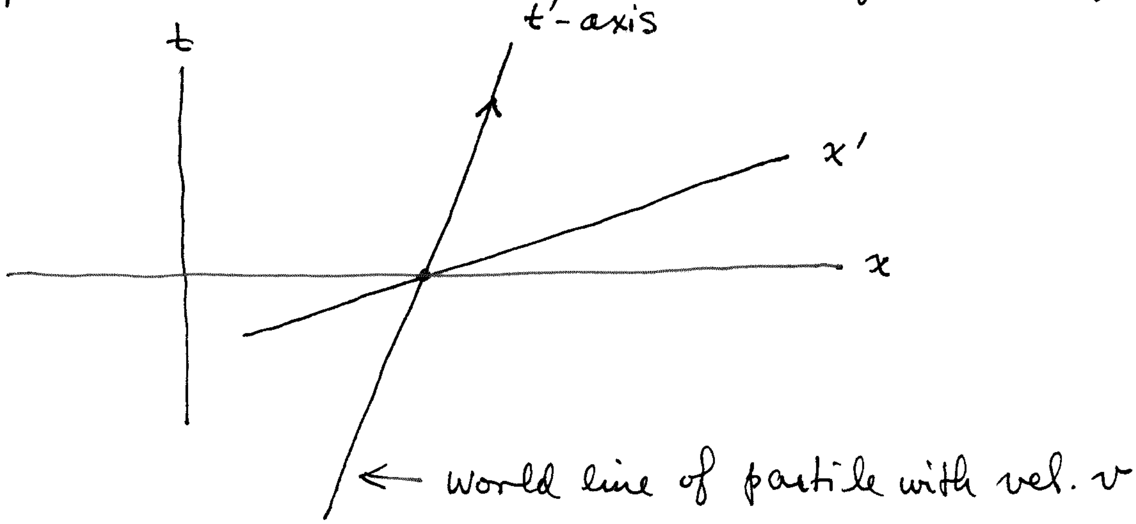
The x-axis is the set of events for which $t=0$. It is the set of events simultaneous with $(0,0)$ in the unprimed frame. Similarly, the x'-axis is the set of events for which $t'=0$, i.e., the set of events simultaneous with the event $(0,0)$ in the primed frame. By the Lorentz transformation, setting $t'=0$, we see that $t=vx$ is the equation of the x'-axis in the unprimed coordinates.



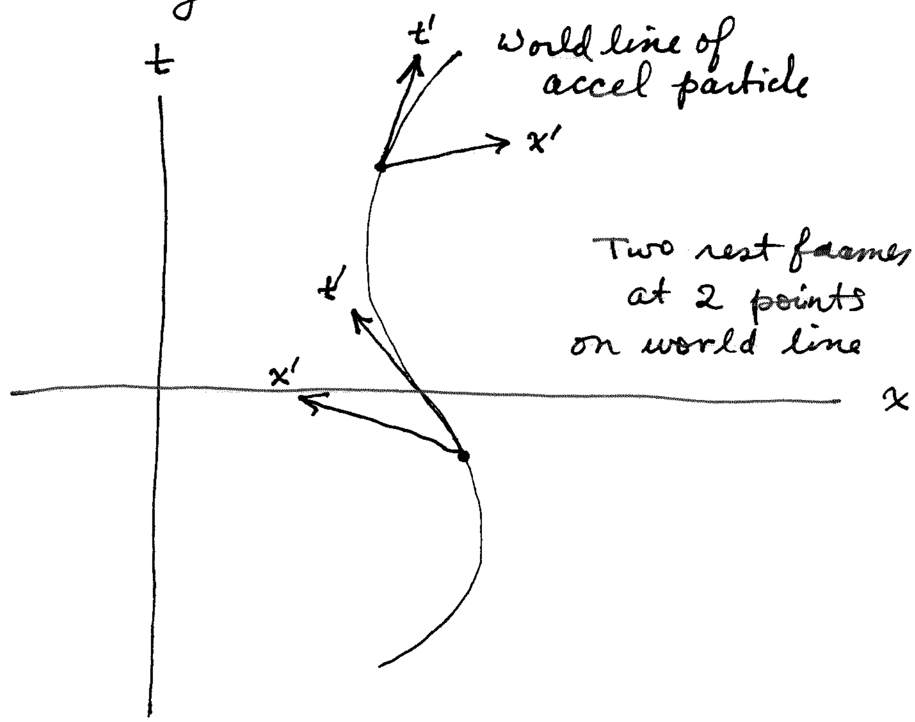
The angles α, α above are equal (in $v=c$ units). The axes of the moving frame collapse around the light cone (dotted line) as $v \to c$.

The x,t axes are orthogonal on the paper but the x',t' axes are not. That is because the Lorentz transformation does not respect the Euclidean geometry of the paper. Instead, it respects a different kind of geometry, as we will see.

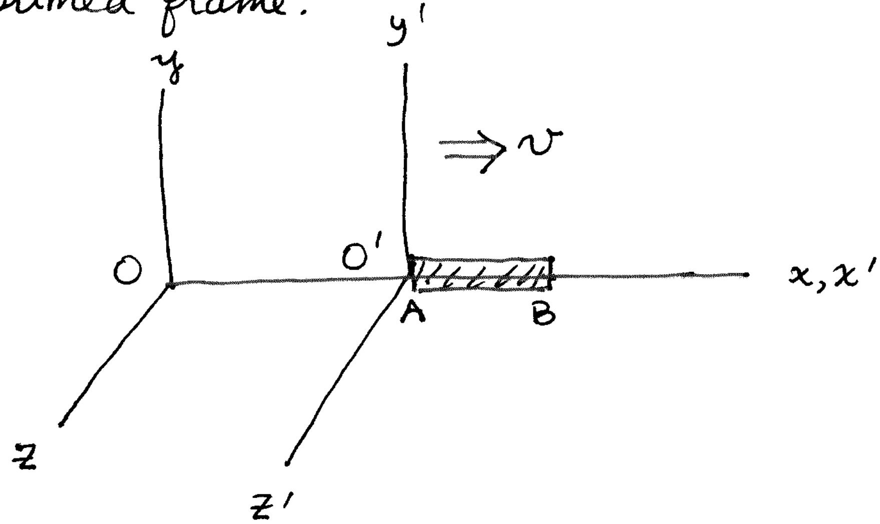
If a particle is moving with constant velocity, then the expression "the rest frame of the particle" refers to a frame in which the particle is stationary. If we move the origin of that frame we can always make the world line of the particle coincide with the t'-axis of the rest frame.



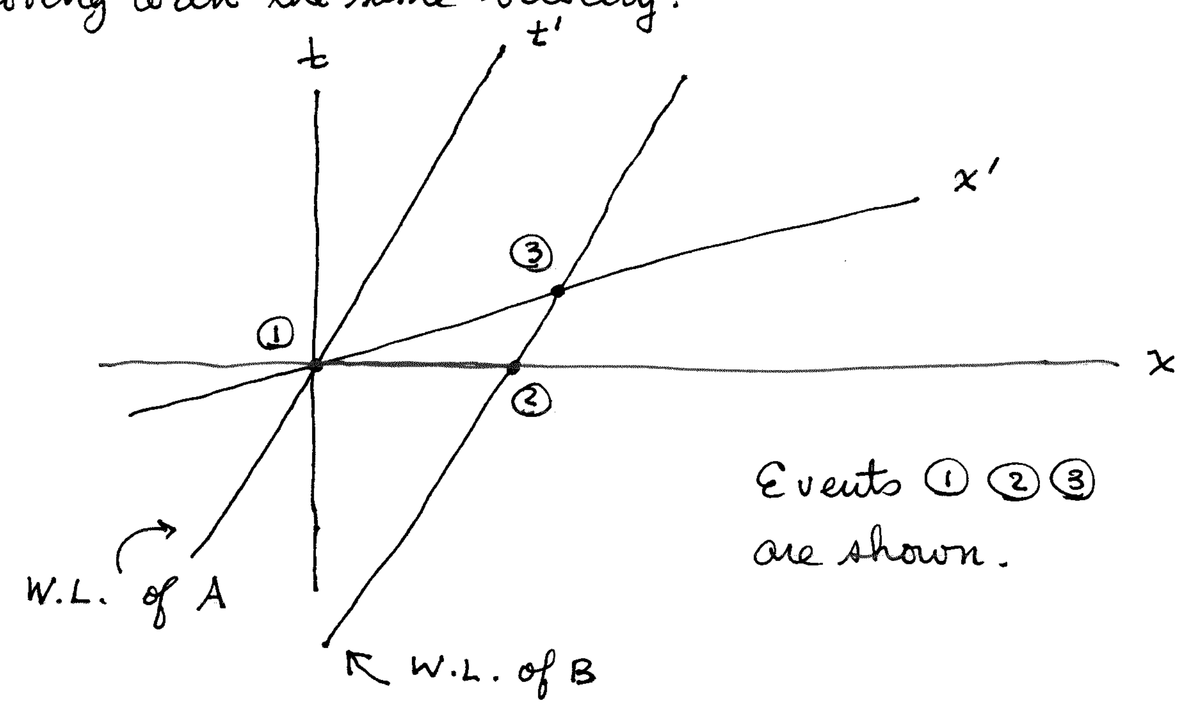
If a particle is accelerated, then its velocity is changing. In that case we can speak of "the rest frame of the particle" at a particular time along the accelerated world line. This rest frame has its t'-axis tangent to the world line. This means that $\vec{v}'$ of the particle is 0 at that point on the world line as measured in that (rest) frame. The particle is instantaneously at rest in its own rest frame.



The Lorentz contraction can be visualized in a space-time diagram. Suppose a rod with ends A,B is stationary in the moving frame. It is moving with velocity v in the unprimed frame.



Consider the world lines of A and B in a space-time diagram. Assuming A is at the origin O' of the moving frame, its coordinate $x' = 0$, so its world line is the t'-axis. World line B is parallel to that, since A, B are moving with the same velocity.



If we measure the length of the rod in some frame, including frames in which the rod is moving, we do so by measuring the positions of A and B at the same time (in the given frame). The measurement events are simultaneous (in the given frame). If we measure the length of the rod in the primed frame, let us call the answer L'. The measurement events are ①, ③ in the diagram above. These events are simultaneous in the primed frame, since $t_1' = t_3' = 0$. Length L' is the "natural" length of the rod, i.e. the length measured in a frame in which the rod is at rest. From the diagram above, $(x_1', t_1') = (0,0)$ and $(x_3', t_3') = (L', 0)$.

Let L be the length of the rod in the unprimed frame. The measurement events are ①, ②, with coordinates $(x_1, t_1) = (0,0)$, $(x_2, t_2) = (L, 0)$.

The eqn of the world line of A (the t'-axis) in the unprimed frame is $x = vt$. The equation of the world line of B in the unprimed frame is $x = vt + L$. In particular, since this world line passes through event ③, we have $x_3 = vt_3 + L$.

Now use the Lorentz transformation on the coordinates of event ③:

$$x_3' = L' = \gamma(x_3 - vt_3) = \gamma L .$$

Thus, $L = \frac{L'}{\gamma}$. The length measured in the unprimed frame is less than the "natural length" L'. The moving rod appears to shrink.

Let us introduce ~~and~~ matrix notation for the Lorentz transformation.

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

This is a "boost" down the x-axis. Now write

$$\left.\begin{aligned} x^0 &= t \\ x^1 &= x \\ x^2 &= y \\ x^3 &= z \end{aligned}\right\} \quad \text{or} \quad x^\mu \text{ collectively, where } \mu = 0, 1, 2, 3.$$

We will write Greek indices μ, ν, α, β, etc. for space-time indices. They run $0, 1, 2, 3$. The upper (superscript) position) on the index μ will be explained later.

Also write $\Lambda^{\mu}{}_{\cdot\nu}$ for the matrix of the Lorentz transformation, where μ is the row index (superscript) and ν is the column index (subscript position). The dot is a place holder, which shows that the μ comes first. The positions of these indices will be explained later.

Then the L.T. is

$$x'^{\mu} = \sum_{\nu} \Lambda^{\mu}{}_{\cdot\nu}\, x^{\nu}.$$

right hand side.

The index ν that is summed over on the R.H.S. is repeated (it occurs twice). ~~the index~~ This index is a dummy index of summation, which can be replaced by any symbol we like ($\alpha, \beta, \sigma, \ldots$). It does not appear on the L.H.S. The index μ on the R.H.S. appears only once. It is not summed over, so it also appears on the L.H.S. About 99% of the formulas in relativity theory have this property: repeated indices are summed and indices that occur only once are not summed.

So to save writing we omit the $\sum_{\nu}$ in the formula above, and <u>assume</u> that <u>all repeated indices are summed</u>. This is called the <u>summation convention</u>. It cannot be used if any index occurs 3 or more times, or if an unrepeated index is summed, etc. Such cases are rare. If they occur, we will note them, and insert explicit summation signs. With the summation convention, the formula above becomes

$$\boxed{x'^{\mu} = \Lambda^{\mu}{}_{\cdot\nu}\, x^{\nu}}$$

Lorentz transformation.

Notice the sum involves one lower and one upper index. This is the usual situation for indices that are summed.

We have written down the L.T. that is a boost in the x-direction. Adding boosts in the y- and z-directions, we have the matrices:

x-boost
$$\begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \Lambda^{\mu}{}_{\nu}$$

coords $x^\mu = (t, \vec{x})$
$= (t, x, y, z)$

y-boost
$$\begin{pmatrix} \gamma & 0 & -\gamma v & 0 \\ 0 & 1 & 0 & 0 \\ -\gamma v & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

z-boost
$$\begin{pmatrix} \gamma & 0 & 0 & -\gamma v \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma v & 0 & 0 & \gamma \end{pmatrix}$$

One can boost in any direction, not only along the x-, y-, and z-axes.

In addition to pure boosts, pure spatial rotations are considered to be Lorentz transformations. These do nothing to the time coordinate, but rotate the spatial axes. The following are the rotation matrices for rotations about the x-, y-, and z-axes by an angle θ:

Rotation about x-axis: by angle θ

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

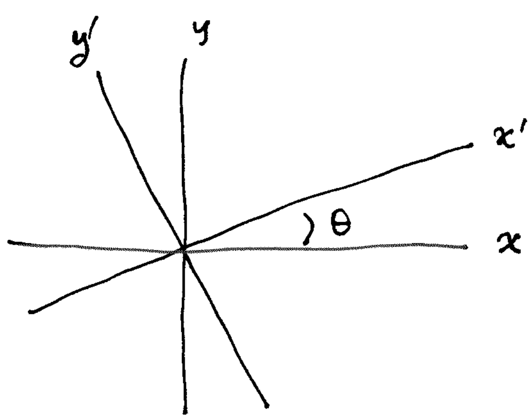
y-axis:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & 0 & -\sin\theta \\ 0 & 0 & 1 & 0 \\ 0 & \sin\theta & 0 & \cos\theta \end{pmatrix}$$

z-axis

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

For example, a rotation about the z-axis means we have old (unprimed or unrotated) coordinates, and new (primed or rotated) coordinates, and the matrix shown expresses the new coordinates as functions of the old:



$$t' = t$$
$$x' = \cos\theta\, x + \sin\theta\, y$$
$$y' = -\sin\theta\, x + \cos\theta\, y$$
$$z' = z$$

Both boosts and rotations affect two of the coordinates and leave the other 2 coordinates alone. For example, a boost down the x-axis affects coordinate t, x, but not y, z; and a rotation about the x-axis affects coordinates y, z, but not t, x. The rotations are expressed in terms of an angle. The boosts are like rotations, too, but the angle is hyperbolic instead of trigonometric. To bring this out, define

$$\gamma = \frac{1}{\sqrt{1-v^2}} = \cosh\lambda$$

$$v\gamma = \frac{v}{\sqrt{1-v^2}} = \sinh\lambda$$

for $-1 < v < +1$, which implies $1 \le \gamma < \infty$. These equations are not independent, since

$$\cosh^2\lambda - \sinh^2\lambda = 1 = \frac{1}{1-v^2} - \frac{v^2}{1-v^2}.$$

Parameter λ is called the <u>rapidity</u>. When $-1 < v < +1$, $-\infty < \lambda < +\infty$. In terms of the rapidity, the Lorentz transformations (the boosts) become:

boost x:
$$\begin{pmatrix} \cosh\lambda & -\sinh\lambda & 0 & 0 \\ -\sinh\lambda & \cosh\lambda & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\text{boost-}y: \begin{pmatrix} \cosh\lambda & 0 & -\sinh\lambda & 0 \\ 0 & 1 & 0 & 0 \\ -\sinh\lambda & 0 & \cosh\lambda & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\text{boost-}z: \begin{pmatrix} \cosh\lambda & 0 & 0 & -\sinh\lambda \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\lambda & 0 & 0 & \cosh\lambda \end{pmatrix}$$

This notation makes the boosts look more like the rotations.

Under a purely spatial rotation, the distance to the origin is invariant, so

$$x^2+y^2+z^2 = x'^2+y'^2+z'^2$$

This distance (squared) has the same expression in the old and new coordinates. Is there a similar invariant under Lorentz transformations?

Yes, it is the <u>Minkowski invariant</u>:

$$t^2-x^2-y^2-z^2 = t'^2-x'^2-y'^2-z'^2 .$$

This is the same expression in any two coordinates on space-time related by a Lorentz transformation.

It is easy to see this in the case of a pure rotation, for which $t'=t$. Then the identity becomes $x^2+y^2+z^2 = x'^2+y'^2+z'^2$, which we know is true for a rotation. Let's check the Minkowski invariant for a boost along the x-axis. Then $y=y'$ and $z=z'$, so we must check $t'^2 - x'^2 = t^2 - x^2$. Just do the algebra:

$$t'^2 - x'^2 = \gamma^2(t^2 - 2vtx + v^2x^2) - \gamma^2(x^2 - 2vtx + v^2t^2)$$

$$= \gamma^2\left[t^2(1-v^2) - x^2(1-v^2)\right] = t^2 - x^2.$$

Similarly we can check the y- and z-boosts.

To express the Minkowski invariant more compactly, we introduce the Minkowski metric tensor. It can be thought of as a matrix,

$$\eta_{\mu\nu} = \begin{pmatrix} +1 & & & \\ & -1 & & 0 \\ & 0 & -1 & \\ & & & -1 \end{pmatrix} \qquad \text{in coords } x^\mu = (t, x, y, z)$$

It has ± 1 on the diagonal, and zeroes off the diagonal. We write $\eta_{\mu\nu}$ with lower indices. The position will be explained later. In terms of $\eta_{\mu\nu}$, the invariant is

$$t^2 - x^2 - y^2 - z^2 = \eta_{\mu\nu}\, x^\mu x^\nu = \eta_{\mu\nu}\, x'^\mu x'^\nu = t'^2 - x'^2 - y'^2 - z'^2$$

Now since μ, ν are dummy indices of summation, we can replace them by α, β and write

$$\eta_{\mu\nu}\, x'^\mu x'^\nu = \eta_{\alpha\beta}\, x^\alpha x^\beta$$

Now use the Lorentz transformation, $\left.\begin{aligned} x'^\mu &= \Lambda^\mu{}_\alpha\, x^\alpha \\ x'^\nu &= \Lambda^\nu{}_\beta\, x^\beta \end{aligned}\right\}$

so,

$$\eta_{\mu\nu}\, x'^{\mu} x'^{\nu} = x^{\alpha} \Lambda^{\mu}{}_{\cdot\alpha}\, \eta_{\mu\nu}\, \Lambda^{\nu}{}_{\cdot\beta}\, x^{\beta} = x^{\alpha} \eta_{\alpha\beta}\, x^{\beta}$$

or, since this is true for all x^{α}, we have

$$\boxed{\Lambda^{\mu}{}_{\cdot\alpha}\, \eta_{\mu\nu}\, \Lambda^{\nu}{}_{\cdot\beta} = \eta_{\alpha\beta}}$$

Let us just write Λ for the matrix $\Lambda^{\mu}{}_{\cdot\alpha}$ (row μ, column α), and η for the matrix $\eta_{\mu\nu}$ (row μ, col ν). Then the eqn. above becomes

$$\boxed{\Lambda^{T} \eta \Lambda = \eta}$$

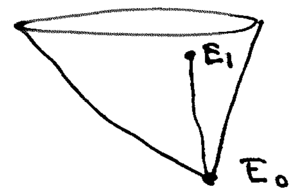
where T means transpose. Compare this to the definition of an orthogonal matrix R,

$$R^{T} I R = I,$$

where I = identity. We see that a Lorentz transformation is orthogonal with respect to (w.r.t.) the Minkowski metric η, while an ordinary orthogonal matrix is orthogonal with respect to the Euclidean metric I.

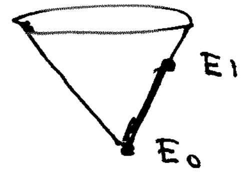
The physical meaning of the Minkowski metric is related to <u>causality</u>. Let's write $x^2+y^2+z^2 = r^2$, the square of the spatial distance from O to (x, y, z). Then the invariant is $t^2 - r^2$, the same in all Lorentz frames. If $t^2 - r^2 > 0$, then $t > r$ and the event $(t\ x\ y\ z)$ lies in the forward light cone (for $t > 0$), and events $E_0 = (0, 0, 0, 0)$ and $E_1 = (t, x, y, z)$ are causally connected. In this case, we say that the space-time interval $E_0 \to E_1$

is <u>time-like</u>



time-like interval,
$t^2 - r^2 > 0$

The interval is also time-like if E_1 lies in the backward light cone. If $t^2 - r^2 = 0$, then E_1 lies on the light cone with vertex at E_0, and we say the interval is <u>light-like</u>:



If $t^2 - r^2 < 0$, then E_1 lies outside the light cone, and we say the interval $E_0 \to E_1$ is <u>space-like</u>.

Since $t^2 - r^2$ is the same in all Lorentz frames, these designations are invariant (all observers agree to them).

$\eta_{\mu\nu}$ (with lower indices) is called the <u>covariant metric tensor</u>. $\eta^{\mu\nu}$ (with upper indices) is called the <u>contravariant</u> metric tensor. It is defined as the inverse of $\eta_{\mu\nu}$, so

$$\eta_{\mu\alpha}\,\eta^{\alpha\nu} = \delta_\mu^\nu = \text{Kronecker } \delta .$$

Now the matrix $\begin{pmatrix} +1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$ is equal to its own inverse, so as a matrix $\eta^{\mu\nu}$ is the same as $\eta_{\mu\nu}$. Nevertheless, the two versions of the metric tensor are used differently and we should keep them distinct. (In fact, the two matrices are not equal in all coordinate systems, just the ones we are currently using.)

Here are some rules of tensor analysis. We will explain them more fully later.

The two metric tensors, $\eta_{\mu\nu}$ and $\eta^{\mu\nu}$, can be used to <u>raise</u> and <u>lower</u> <u>indices</u>. Suppose A^μ is a quantity with an upper (or <u>contravariant</u>) index. Then we define a quantity A_μ (with a lower, or <u>covariant</u>, index) by

$$A_\mu = \eta_{\mu\nu} A^\nu .$$

e.g. A

We usually use the same symbol for the contravariant and covariant versions, because physically they represent the same thing (for example, the electromagnetic vector potential).

For example, we know $x^\mu = (t, \vec{x})$, so

$$x_\mu = (t, -\vec{x})$$

Lowering the index just changes the sign of the spatial part of a 4-vector.

Similarly, let B_μ be a quantity (a <u>covariant</u> <u>vector</u>) with a lower or covariant index. Then we define

$$B^\mu = \eta^{\mu\nu} B_\nu$$

This creates a <u>contravariant</u> <u>vector</u> B^μ.

Here's an example of raising and lowering indices. Take the basic property of a Lorentz transformation:

$$\Lambda^{\mu}_{\cdot\alpha}\ \eta_{\mu\nu}\Lambda^{\nu}_{\cdot\beta} = \eta_{\alpha\beta}$$

and write $\Lambda_{\mu\beta} = \eta_{\mu\nu}\Lambda^{\nu}_{\cdot\beta}$ (lower the first index on $\Lambda^{\nu}_{\cdot\beta}$)
so

$$\Lambda^{\mu}_{\cdot\alpha}\ \Lambda_{\mu\beta} = \eta_{\alpha\beta}.$$

Now multiply by $\eta^{\beta\sigma}$ and use fact that η^{xx} is the inverse of η_{xx} :

$$\Lambda^{\mu}_{\cdot\alpha}\Lambda_{\mu\beta}\,\eta^{\beta\sigma} = \eta_{\alpha\beta}\,\eta^{\beta\sigma}$$

or $\boxed{\Lambda^{\mu}_{\cdot\alpha}\ \Lambda_{\mu}^{\cdot\,\sigma} = \delta^{\sigma}_{\alpha}}$ (another way to write the basic property of Λ.) Here δ^{σ}_{α} is the Kronecker δ. All explicit reference to η has disappeared. This formula is useful because it shows how to invert a Lorentz transformation: Lower the first index and raise the second, then take the transpose.

Note: We began by defining a Lorentz transformation as a boost down the x-axis, then extended that to other boosts and proper rotations and all transformations that can be generated by composing such transformations. Another point of view is to say that a Lorentz transformation is any transformation that satisfies the boxed equation above. Are the two definitions the same? Not quite — the first gives us the proper Lorentz transformations, while the second includes also time reversal and parity. (Just a remark.)

Here is another application of raising and lowering indices:

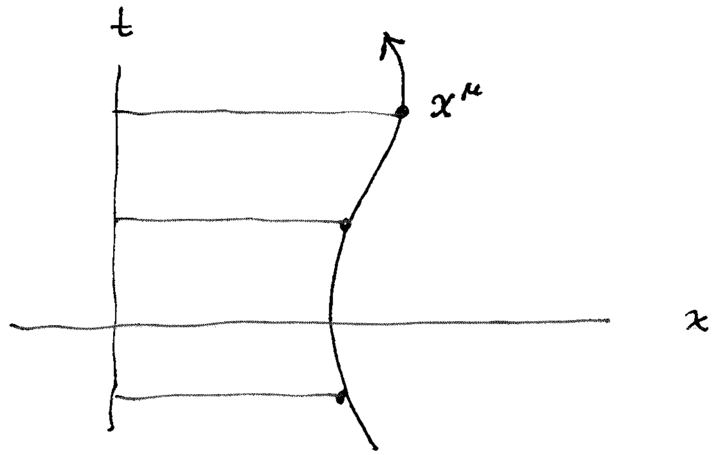
The (Minkowski) invariant is

$$x^\mu \eta_{\mu\nu} x^\nu = x^\mu x_\mu = t^2 - |\vec{x}|^2$$

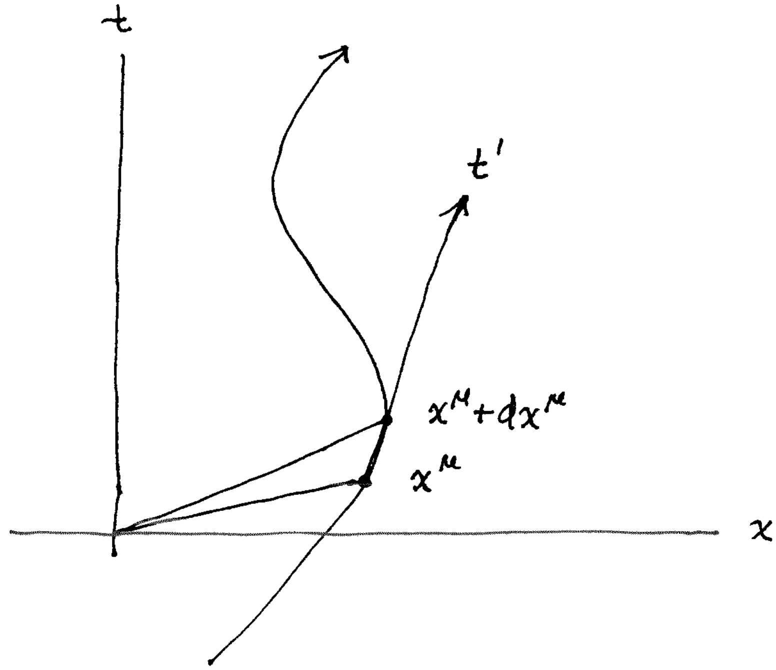
where $x^\mu = (t, \vec{x})$, $x_\mu = (t, -\vec{x})$. Notice we sum over one contravariant and one covariant index to get an invariant.

Let's return to an accelerated particle. We wish to find a convenient parameter to describe the world line of the particle. One possibility is just to use t, that is, to write $x^\mu(t)$. This just means

$$t = t$$
$$\vec{x} = \vec{x}(t)$$



so the $x^0 = t$ component is trivial, and the spatial part is just how we would describe a trajectory in nonrelativistic mechanics. But t is the time coordinate associated with a particular reference frame, and it would be desirable to have a parameterization that is independent of the observer. So consider two nearby points (really events) on the world line, x^μ and $x^\mu + dx^\mu$.



x' axis not shown

The displacement dx^μ is tangent to the world line, and so is along the t-axis of the instantaneous rest frame of the particle. Call the rest frame the primed frame. Now look at the invariant associated with the interval dx^μ:

$$dx^\mu dx_\mu = dt^2 - |d\vec{x}|^2 = dt'^2 - |d\vec{x}\,'|^2$$

It is the same in all frames. But the particle is at rest in its own rest frame, so $d\vec{x}\,' = 0$ and

$$dt'^2 = dx^\mu dx_\mu = dt^2 - |d\vec{x}|^2.$$

Physically, dt' is the amount of elapsed time ticked off by a clock carried with the particle. ~~this~~ Let us call this time τ (instead of t'), so

$$d\tau^2 = dt^2 - |d\vec{x}|^2.$$

We call τ the <u>proper time</u>, and use it as a parameter of the world line, $x^\mu = x^\mu(\tau)$. This is the most convenient parameterization of a world line in relativity theory.

τ increases along the world line, so $d\tau/dt > 0$. In fact,

$$d\tau = \sqrt{dt^2 - |d\vec{x}|^2}$$

$$\frac{d\tau}{dt} = \sqrt{1 - \left|\frac{d\vec{x}}{dt}\right|^2} = \sqrt{1-v^2} = \frac{1}{\gamma}.$$

So, $d\tau = \frac{dt}{\gamma}$, the clock carried by the particle runs slower by a factor of γ. This is the <u>time-dilation</u>.

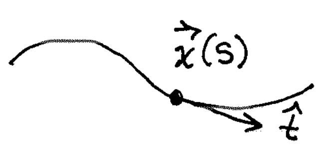
The derivative

$$\frac{dx^\mu}{d\tau} = \begin{pmatrix} \frac{dt}{d\tau} \\ \frac{d\vec{x}}{d\tau} \end{pmatrix}$$

is called the <u>world velocity</u> of the particle. It is a unit vector in Minkowski geometry,

$$\frac{dx^\mu}{d\tau}\frac{dx_\mu}{d\tau} = \frac{d\tau^2}{d\tau^2} = 1.$$

Parameterizing a world line by τ is similar to parameterizing an ordinary curve in 3D space by the arc length s,



where $ds^2 = |d\vec{x}|^2$, so $\hat{t} = \frac{d\vec{x}}{ds}$ is the unit tangent vector along the curve.

If we multiply the world velocity by the mass of the particle, we get the <u>4-momentum</u>:

$$p^\mu = m \frac{dx^\mu}{d\tau} .$$

It has the invariant square,

$$p^\mu p_\mu = m^2 \frac{dx^\mu}{d\tau} \frac{dx_\mu}{d\tau} = m^2 .$$

It is related to the mechanical energy and momentum by

$$p^\mu = \begin{pmatrix} E \\ \vec{p} \end{pmatrix} = m \begin{pmatrix} \frac{dt}{d\tau} \\ \frac{d\vec{x}}{d\tau} \end{pmatrix} = m \begin{pmatrix} \frac{dt}{d\tau} \\ \frac{d\vec{x}}{dt} \frac{dt}{d\tau} \end{pmatrix} = \begin{pmatrix} m\gamma \\ m\gamma\vec{v} \end{pmatrix} .$$

If we restore the c's, this is

$$\left. \begin{aligned} E &= mc^2\gamma \\ \vec{p} &= m\gamma\vec{v} \end{aligned} \right\} .$$

the usual expressions for energy and momentum in relativity. The energy is the kinetic energy plus the rest mass/energy, mc^2. The relation $p^\mu p_\mu = m^2$ is

$$E^2 - |\vec{p}|^2 = m^2 ,$$

or $\quad E = \sqrt{m^2 + |\vec{p}|^2}$. This is $E = \sqrt{m^2c^4 + c^2p^2}$

in ordinary units.