Physics H190
Spring 2013
Homework 4
Due Wednesday, February 27, 2013

Reading Assignment: My lecture last Wednesday followed the lecture notes for Wednesday, February 13 on the quantization of the vibrating string and the infinite zero-point energy. There is also some discussion of that subject in chapter 1 of the book. Some additional discussion was given in class. Please review that material. We will return to the zero point energy later, in connection with the Casimir effect.

In dealing with the vibrating string, be careful of the notation. At one point I called the mode number n, but I later switched to m because n got used for the quantum number of a harmonic oscillator. So m in this context is not the mass. The role of the mass of a mechanical harmonic oscillator is played by μ, the mass per unit length, of the vibrating string. Obviously the units are not the same, but when the units of the other variables in the equations are taken into account (the q's and p's), the Hamiltonian still has dimensions of energy.

I would like you to be comfortable with the idea that a free quantum field is represented by an infinite collection of harmonic oscillators, one for each normal mode of the classical field.

1. Suppose the sum for the zero point energy of the vibrating string is cut off at a mode number for which the wave length is equal to the Planck length. The Planck length is the unique length that can be constructed out of G (Newton's constant of gravitation), $\hbar$ and c. You may assume that the mode number at which this cutoff occurs is very large. Show that the zero point energy of the string is proportional to its length (called L in the notes). Evaluate this numerically (ergs/cm, or Joules/meter), assuming that the phase velocity of the string v is the speed of light c. This is not a very realistic assumption for a real vibrating string, where we would have $v \ll c$, but we are using the string as a crude model of real quantum fields (e.g., the electromagnetic), where $v = c$. Then using $E = mc^2$, compute the mass per unit length (grams/cm or kg/meter).

2. In classical mechanics, the *configuration* of a system is enough information to specify the positions of all the particles. For example, in the helium atom, the configuration consists of $(\mathbf{x}_1, \mathbf{x}_2)$, the positions of the two electrons. In ordinary quantum mechanics, the wave function is a complex-valued function of the classical configuration. For example, in the helium atom, it is $\psi(\mathbf{x}_1, \mathbf{x}_2)$ (ignoring the spin of the electrons).

In field theory, the wave function can be regarded as a complex-valued functional of the classical configuration. For example, for the vibrating string, the classical configuration is specified by the function $y(x)$ (at a given time), so the wave functional is $\Psi[y(x)]$.

(a) Let m be a mode index for the vibrating string (in some parts of the notes this may be denoted

n instead of m). Express the mode amplitude q_m in terms of the shape of the string, $y(x)$. Notice that q_m is a functional of $y(x)$.

(b) Let the Hamiltonian for a one-dimensional harmonic oscillator be

$$H = \frac{p^2}{2\mu} + \frac{\mu\omega^2}{2}x^2.$$

Write down the ground state wave function $\psi(x)$. You can refer to your quantum text book, all you have to do is possibly change notation.

Now consider a two-dimensional harmonic oscillator,

$$H = \frac{p_x^2}{2\mu} + \frac{p_y^2}{2\mu} + \frac{\mu\omega_x^2}{2}x^2 + \frac{\mu\omega_y^2}{2}y^2.$$

Notice that the frequencies of the x- and y-oscillators are allowed to be different. Write down the ground state wave function $\psi(x, y)$ of this oscillator.

(c) Consider the quantized vibrating string. The wave function is

$$\Psi(q_1, q_2, \ldots),$$

a function of the infinite set of q_m's. Since each q_m is a functional of $y(x)$, Ψ is also (implicitly) a functional of $y(x)$. Write down an expression for the ground state wave function of the vibrating string. Compare to formula (1.9) in Mukhanov and Winitzki.

3. Send me a question, at the h190 email address (not my regular email account).

More chapters of Mukhanov and Winitski will be posted soon.