Physics H190
Spring 2005
Homework 3
Due Wednesday, February 16, 2005

The subject of electric and magnetic energy in the presence of matter (dielectrics and magnetic materials) is one that has confused generations of physicists. One reason is that many books treat this subject incorrectly, including Griffiths and Jackson. I suspect Landau and Lifshitz (*The Electrodynamics of Continuous Media*) do it right, but I haven't checked. Where the books go wrong is that they try to analyze the energy balance without regard for thermodynamic considerations. See Griffiths, *Introduction to Electrodynamics*, 3rd ed., p. 191, or Jackson, *Classical Electrodynamics*, 3rd ed., p. 165.

In this problem we concentrate on the electric case, because the analysis is somewhat simpler than the magnetic case, but the practical applications tend to be magnetic. Even if not practical, the electric case is good practice before trying to understand the magnetic case. First some background from electrostatics.

If you have a piece of dielectric, and you bring in some charge from infinity into the neighborhood, then the electric field of the charge polarizes the molecules in the dielectric and creates a nonzero polarization $\mathbf{P}$ (defined as the dipole moment per unit volume) inside the dielectric. See Fig. 1. The polarization creates its own electric field that adds to the field of the external charge, creating a total electric field. The individual dipoles that create $\mathbf{P}$ respond to the total electric field, so it may not be an easy problem to calculate $\mathbf{P}$, given the external charge distribution, since the solution must be self-consistent. Nevertheless, given some external (or "free") charge distribution, there is some definite $\mathbf{P}$ set up inside the matter and some definite total $\mathbf{E}$ (electric field) that results everywhere in space.

The electric field produced by the dipoles can be computed if $\mathbf{P}$ is given. It is the field produced by the "bound" charged distributions, of which there is a volume distribution ρ_b and a surface distribution σ_b (that is, at the surface of the dielectric), where $\rho_b = -\nabla \cdot \mathbf{P}$ and $\sigma_b = \hat{\mathbf{n}} \cdot \mathbf{P}$, where $\hat{\mathbf{n}}$ is the outward normal to the surface.

A linear dielectric is one in which $\mathbf{P}$ is proportional to $\mathbf{E}$ (this is the total electric field, not just that produced by the external charges). In SI units, we write

$$\mathbf{P} = \epsilon_0 \chi \mathbf{E}, \tag{1}$$

where χ is the electric susceptibility.

For any dielectric, linear or nonlinear, we define the displacement by

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}, \tag{2}$$

which implies the Maxwell equation,

$$\nabla \cdot \mathbf{D} = \rho_f, \tag{3}$$

where ρ_f is the charge density of the external or free charges.

A standard analysis presented (correctly) in all the books concerns the amount of work δW that must be done to bring up some additional increment in free charge $\delta\rho_f$, given some ρ_f that is already present and the fields $\mathbf{P}$, $\mathbf{E}$ and $\mathbf{D}$ that are produced by it. This work is

$$\delta W = \int d^3\mathbf{x}\, \delta\rho_f(\mathbf{x}) \Phi(\mathbf{x}), \tag{4}$$

where Φ is the electrostatic potential, so $\mathbf{E} = -\nabla\Phi$. This is just simple electrostatics. Now using $\nabla \cdot \mathbf{D} = \rho_f$ and hence $\nabla \cdot \delta\mathbf{D} = \delta\rho_f$, we can use integration by parts to turn Eq. (4) into

$$\delta W = \int d^3\mathbf{x}\, \mathbf{E} \cdot \delta\mathbf{D}. \tag{5}$$

This is also done correctly in all the books.

All we have done is compute the work required to change the external charge distribution by a small amount. You might be more interested in finding the total work required to bring in the whole external charge distribution, not just a small increment. The books correctly point out that this is a hard problem in general, especially for nonlinear dielectrics, because of the difficulty of the self-consistent calculation of $\mathbf{P}$. So, they say, let's look at linear dielectrics.

Here is what they now say. For a linear dielectric, Eqs. (1) and (2) imply

$$\mathbf{D} = \epsilon \mathbf{E}, \tag{6}$$

where

$$\epsilon = \epsilon_0 (1 + \chi). \tag{7}$$

Thus, $\mathbf{D}$, $\mathbf{E}$ and $\mathbf{P}$ are all proportional to each other, and, in particular,

$$\delta\mathbf{D} = \epsilon \delta\mathbf{E}. \tag{8}$$

Thus,

$$\delta W = \int d^3\mathbf{x}\, \epsilon \mathbf{E} \cdot \delta\mathbf{E} = \delta \int d^3\mathbf{x}\, \frac{\epsilon}{2} E^2, \tag{9}$$

or, assuming that $W = 0$ when $\mathbf{E} = 0$,

$$W = \int d^3\mathbf{x}\, \frac{\epsilon}{2} E^2 = \frac{1}{2} \int d^3\mathbf{x}\, \mathbf{E} \cdot \mathbf{D}. \tag{10}$$

Precisely speaking, this result is incorrect, because when you bring in some small increment in external charge $\delta\rho_f$, you change the thermodynamic state of the dielectric, in particular, its temperature. Since the susceptibility χ or dielectric constant ϵ are in general functions of temperature, they also change. Thus, one must add a term $(\delta\epsilon)\mathbf{E}$ to Eq. (8), and one can no longer add up the small increments to get the answer (10). Alternatively, one could hold the dielectric in contact with a heat reservoir so its temperature (and hence χ and ϵ) remains constant, but then when you bring in the external charge, heat flows into or out of the system, and this heat must be included in the energy balance.

1. In this problem we consider a parallel plate capacitor. See Fig. 2. We assume that the spacing between the plates is small compared to the linear dimensions of the plates themselves, because this allows us to ignore fringe fields. Thus, by symmetry, $\mathbf{E}$, $\mathbf{P}$ and $\mathbf{D}$ are all in the same direction (up, in the figure), and we can describe them by their magnitudes E, P and D. A certain free charge density σ_f is spread uniformly over the lower plate, with and equal and opposite charge density on the upper plate. By Gauss' law, $\sigma_f = D$. Notice that $\sigma_f = D$ is a parameter under experimental control. Φ is the potential difference between the upper and lower plate. Since $\mathbf{P}$ is uniform in the dielectric, the bound volume charge density ρ_b vanishes, but the bound surface charge density is nonzero. The bound surface charge layer lies adjacent to the free charge layer and partially cancells it, reducing the electric field and hence the potential between the plates below the value it would have (for equal σ_f) in the absence of a dielectric.

We take the dielectric to be a polar gas, such as HCl, in which the molecules are small dipoles. We do not assume that the response is linear, meaning that electric fields may be large or temperatures low. The polar molecules tend to align in the electric field, but thermal agitation tends to knock them out of alignment, so for fixed electric field, the net alignment (hence $\mathbf{P}$) is a decreasing function of temperature. The average polarization of a dipole in a given electric field can be computed by statistical mechanics, which gives the Langevin equation, but we will not need this.

In class and the notes we have considered this system, and we decided that the first law of thermodynamics takes the form,

$$dQ = dU + P\,dV - VE\,dD,$$

for infinitesimal, reversible processes. Don't confuse the pressure P with the polarization P (easy, since in this problem we don't use the pressure). Here V is the volume of the dielectric, which in this problem will not change (hence the $P\,dV$ term will be zero). The last term is the same as Eq. (5), with the integral carried out over the volume of the dielectric, but with an opposite sign since in thermodyanamics we talk about the work done *by* the system, not the work done *on* the system.

(a) Suppose the charge on the plates is held fixed, but the the dielectric is heated. Does E increase or decrease? Use physical reasoning to get your answer.

(b) Suppose the dielectric is thermally isolated, and some extra charge is added to the plates (hence D increases). Does the temperature of the dielectric go up or down? Hint: Consider cross derivatives of the free energy $F = U - TS$ to get the answer. I was not able to justify this answer by any obvious "physical" reasoning, but perhaps you can do better.

(c) In this part assume for simplicity that the dielectric is linear. Consider an initial state in which there is some charge on the dielectric. Plot this as a point in the E-D plane. Sketch an isothermal curve starting at this point, as D is increased. Then sketch other isothermal and/or adiabatic curves, forming a Carnot cycle. The cycle should convert thermal energy into electrical energy (charge is placed on the plates at one potential, and removed at a higher potential).