Physics 250
Fall 2015
Notes 2
Differential Geometry of Lie Groups

1. Introduction

These notes concern the differential geometry of Lie groups. This is a large and rich subject, and we provide only an introduction to some of the main ideas.

2. Group Actions

We begin with some material on *group actions*, an extremely useful concept. Most of this has already been discussed in various homeworks.

Let X be a space (a set of points of any kind). Let $\mathrm{Bij}(X)$ be the set of all bijections of X onto itself. Such a bijection can be thought of as a permutation of the points of the set X. The set $\mathrm{Bij}(X)$ forms a group under the composition. If $X = M$ is a differentiable manifold, we may wish to consider instead the space $\mathrm{Diff}(M)$, the space of all diffeomorphisms of M onto itself, which also forms a group. Sometimes we refer to the elements of $\mathrm{Bij}(X)$ or $\mathrm{Diff}(M)$ as *transformations* of X or M (Lorentz transformations, canonical transformations, unitary transformations, etc).

Now let G be a group. A homomorphism $: G \to \mathrm{Bij}(X)$ is said to be an *action* of G on X. We often denote the action by $g \mapsto \Phi_g$, where $\Phi_g : X \to X$ is a bijection, and where

$$\Phi_g \Phi_h = \Phi_{gh}. \tag{1}$$

The transformations Φ_g reproduce the group multiplication law.

In a sense, G is the "abstract" group (a set of objects that obey the group multiplication law, but have no other properties), and the set $\{\Phi_g\}$ is the "concrete" group (a set of objects that obey the group multiplication law, but have other properties as well). In the case that $X = V$ is a vector space and the transformations $\Phi_g : V \to V$ are linear, the action of G on V is called a *representation*.

3. Orbits of a Group Action

Let G act on X, and let $x \in X$. Then the set,

$$\{\Phi_g x | g \in G\}, \tag{2}$$

is called the *orbit* of the point x under the action $g \mapsto \Phi_g$. The orbit of x is the set of all points in X that can be reached from x by applying group operations. The orbit can consist of discrete points, a smooth submanifold (if X is a manifold), or other possibilities, depending on G, X, and the action. As an example, think of the action of $SO(3)$ on $\mathbb{R}^3$; the orbit of a vector $x \in \mathbb{R}^3$ is a sphere (a 2-dimensional surface) if $x \neq 0$, otherwise it is a single point.

Don't confuse this (mathematical) use of the word "orbit" with an orbit in the sense in classical mechanics. However, an orbit in phase space in classical mechanics actually is an orbit in the mathematical sense. A point x in phase space contains the initial positions and momenta of all the particles, and the time-advance map Φ_t applied to this point, for $t \in \mathbb{R}$, generates the orbit in the sense of classical mechanics. It is also the orbit of the group action of $\mathbb{R}$ on phase space, that is, $t \mapsto \Phi_t$.

Given an action of G on X, obviously every point of X belongs to some orbit. Moreover, the individual orbits are disjoint. Therefore a group action divides the space X into mutually disjoint subsets. The points on a given orbit can be thought of as belonging to an equivalence class (points $x, y \in X$ are equivalent if $x = \Phi_g y$ for some $g \in G$). With this understanding, we can write the orbit itself as $[x]$ (using a representative element to define the set).

4. The Isotropy Subgroup

Given a point $x \in X$, we define

$$I_x = \{g \in G | \Phi_g x = x\}, \tag{3}$$

called the *isotropy subgroup* or *stabilizer* of x under the group action. It is the set of all group elements that leave x invariant. The isotropy subgroup actually is a subgroup of G (as you can easily show). The isotropy subgroup I_x may depend on x (it is generally different for different points x). For example, in the action of $SO(3)$ on $\mathbb{R}^3$, the isotropy subgroup of any nonzero vector x is the $SO(2)$ subgroup of rotations about the axis defined by x, whereas if x is the zero vector, then it is the whole group $SO(3)$. Extreme cases of an isotropy subgroup are $I_x = \{e\}$, in which case every group element except the identity does something to x, and $I_x = G$, in which case all group elements leave x invariant. In the latter case, we say that x is a *fixed point* of the group action.

In the case $I_x = \{e\}$ it is possible to label the points of an orbit $[x]$ by group elements, that is, we assign y the label g if $y = \Phi_g x$. In this case, we have a one-to-one correspondence between points of G and points of the orbit $[x]$. If various smoothness and niceness conditions are met, then G and $[x]$ are diffeomorphic. Notice that the point x itself is assigned the identity element e; effectivly x plays the role of an "origin" in the orbit. However, since x was just a representative element, so is the "origin"; by choosing a different representative element in place of x, we obtain a different labeling of the points of the orbit by group elements.

In the general case (I_x is any subgroup of G), the set of group elements that map x to some given $y \in [x]$ is a coset of I_x in G. That is, if

$$y = \Phi_g x = \Phi_{g'} x, \tag{4}$$

then

$$\Phi_{g^{-1}} \Phi_{g'} x = x, \tag{5}$$

so $g^{-1}g' \in I_x$, or $g' = ga$ where $a \in I_x$. This means that g and g' belong to the same left coset of I_x in G. Thus, the points of the orbits are placed into one-to-one correspondence with the cosets of I_x in G. If all spaces are manifolds and all maps smooth, then the orbit $[x]$ is diffeomorphic to the coset space, G/I_x. For example, consideration of the orbits of $x \neq 0$ in $\mathbb{R}^3$ under the action of $SO(3)$ shows that

$$\frac{SO(3)}{SO(2)} = S^2. \tag{6}$$

If points $x, y \in X$ belong to the same orbit, then I_x and I_y are conjugate subgroups in G, that is, there exists some $g \in G$ such that $I_y = gI_xg^{-1}$. Conjugate subgroups are isomorphic, and in particular have the same number of elements.

5. Terminology for Group Actions

Here is some terminology regarding group actions. The definitions can be written in several equivalent forms. All three definitions that follow refer to an action $g \mapsto \Phi_g$ of a group G on a space X.

The action is *transitive* if X consists of a single orbit, that is, if every point of X can be reached from every other point by applying some group operation.

The action is *free* if all transformations except $\Phi_e = \mathrm{id}_X$ move all points of X. That is, the action is free if $I_x = \{e\}$ for all $x \in X$. That is, the action is free if every orbit can be placed in one-to-one correspondence with G (the orbits are "copies" of G, or diffeomorphic to G in the case that everything is smooth).

The action is *effective* if all transformations except $\Phi_e = \mathrm{id}_X$ move some point of X. That is, the action is effective if the kernel of the action $G \to \mathrm{Bij}(X)$ is the trivial subgroup $\{e\}$, that is, if the mapping : $G \to \{\Phi_g | g \in G\}$ is an isomorphism.

6. Actions of G on Itself

An arbitrary group G has an action on itself by left and right translations. Let $a \in G$, and define

$$L_a : G \to G : g \mapsto ag, \tag{7a}$$

$$R_a : G \to G : g \mapsto ga, \tag{7b}$$

where L_a and R_a are called left and right translations, respectively. The mapping $a \mapsto L_a$ is an action. The mapping $a \mapsto R_a$ is not a (left) action, but $a \mapsto R_{a^{-1}}$ is.

A third action of a group on itself is given by $a \mapsto I_a$, where $I_a : G \to G$ is the "inner automorphism"

$$I_a g = aga^{-1}. \tag{8}$$

In other words,

$$I_a = L_a R_{a^{-1}} = R_{a^{-1}} L_a. \tag{9}$$

Nakahara (p. 224) denotes I_a by ad_a and calls it the "adjoint representation." As far as I know, this terminology is not generally used. The word "representation" is usually used for the action of a group on a vector space by linear transformations. The maps I_a are not linear, and G is not a vector space. There is something called the "adjoint representation," to be defined later. It is indeed a linear action on a vector space, as we shall see.

In general (for a non-Abelian group), operations L_a and L_b do not commute, $L_a L_b \neq L_b L_a$, and similarly $R_a R_b \neq R_b R_a$. But left and right translations always commute, $L_a R_b = R_b L_a$, for all $a, b \in G$. If G is Abelian, then $L_a = R_a$, and $I_a = \mathrm{id}_G$ ($a \mapsto I_a$ is a trivial action).

7. Notation for Tangent Maps

We now develop some notation to distinguish the tangent map at a point from the tangent map acting on vector fields.

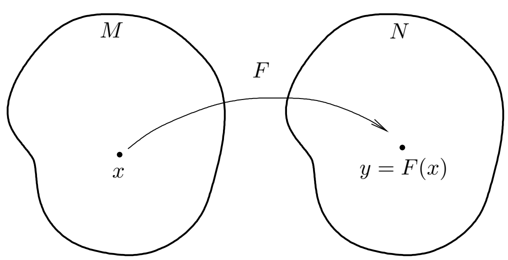

Fig. 1. A map F between two manifolds, $F : M \to N$.

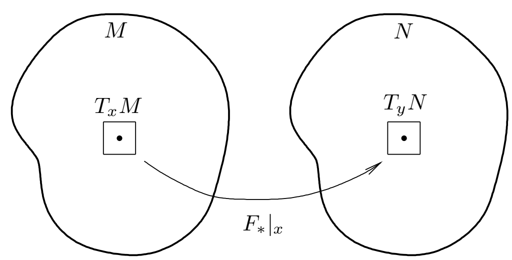

Fig. 2. The tangent map $F_*|_x$ is a linear map between T_xM and T_yN, where $y = F(x)$. There is one such map for each $x \in M$.

Let $F : M \to N$ be a smooth map between manifolds, let $x \in M$ and let $y = F(x) \in N$ (see Fig. 1). Then the tangent map at x is a linear map between the tangent spaces at $x \in M$ and $F(x) \in N$:

$$F_*|_x : T_xM \to T_{F(x)}N \tag{10}$$

(see Fig. 2). Although in some places we might write simply F_* for the tangent map, really there is a different map for each point $x \in M$, so here we are adding "$|_x$" to indicate which tangent map is intended. In this usage, there is no restriction on the map F, and, in particular, the manifolds M and N may have different dimensions.

If, however, F is a diffeomorphism (in which case $\dim M = \dim N$), then the tangent map can be used to map vector fields on M into vector fields on N,

$$F_* : \mathfrak{X}(M) \to \mathfrak{X}(N), \tag{11}$$

where we omit the "$|_x$" designation. The relation between the notations of Eq. (10) and (11) is

$$(F_*|_x)(X|_x) = (F_*X)|_{F(x)}, \tag{12}$$

where $F : M \to N$ is a diffeomorphism, where $X \in \mathfrak{X}(M)$, and where $X|_x$ means X evaluated at $x \in M$, that is, $X|_x \in T_xM$ (in other places in this course we might use notation such as $X(x)$ instead of $X|_x$).

8. Lie Groups

Now we begin the differential geometry of Lie groups. A Lie group is a group that is also a manifold, in which the operations of multiplication and taking the inverse are smooth. The group axioms endow a group manifold with a certain geometrical structure. First, we note that a group has a privileged point e, the identity.

The group manifold may consist of more than one connected component. For example, $O(3)$ consists of two components, the proper and improper rotations (distinguished by $\det R = \pm 1$), and the Lorentz group consists of four components. The component containing the identity is called the *identity component*, and it is a subgroup of the full group. By restricting consideration to the identity component, we have a group manifold that is connected.

The groups commonly encountered in physics, such as $SU(2)$ and $SO(3,1)$ (the spin rotation group and Lorentz group, respectively) are matrix groups, defined by a set of constraints on (usually) real or complex matrices. For example, $SU(2)$ consists of 2×2 complex matrices u that satisfy $uu^\dagger = 1$ and $\det u = 1$. The set of matrices of a finite dimension can be seen as a vector space, for example, the set of all 2×2 complex matrices is the vector space $\mathbb{C}^4 = \mathbb{R}^8$, where the the real coordinates are the real and imaginary parts of the complex coordinates. The points (that is, matrices) of this space can be multiplied with one another, so this space has more structure than a bare vector space. However, it is not a group, for example, it contains the zero matrix. The constraints that are used to define a group such as $SU(2)$ can be seen geometrically as the specification of a submanifold of matrix space. For example, an easy dimension count of the constraints involved in the definition of $SU(2)$ shows that it is a 3-dimensional submanifold of matrix space $\cong \mathbb{R}^8$. (Here $\cong$ mean, "is diffeomorphic to.") In fact, we know from earlier lectures that $SU(2) \cong S^3$. In a similar manner the Lorentz group can be seen as a 6-dimensional submanifold of the matrix space of 4×4 real matrices ($\cong \mathbb{R}^{16}$).

9. Left- and Right-Invariant Vector Fields

Next we consider vector fields on G. The space of such vector fields, $\mathfrak{X}(G)$, is an infinite-dimensional space, but there are certain privileged vector fields of special interest. These are the *left-invariant vector fields* (LIVF's) and *right-invariant vector fields* (RIVF's), defined respectively by

$$L_{a*}X = X, \tag{13a}$$

$$R_{a*}X = X, \tag{13b}$$

for all $a \in G$, where $X \in \mathfrak{X}(G)$. Everything that can be done with LIVF's can also be done with RIVF's, so for now we concentrate on LIVF's. A LIVF can also be defined by

$$(L_{a*}|_g)(X|_g) = X|_{ag}, \tag{14}$$

for all $a, g \in G$, which comes from evaluating both sides of Eq. (13a) at ag and using Eq. (12). When we map the vector $X|_g$ attached to point $g \in G$ using L_{a*}, we get a new vector attached to point ag, which, if X is a LIVF, equals the vector field X evaluated at that point. Intuitively, the mapping L_{a*} proceeds by acting on both the base and the tip of the infinitesimal arrow by L_a (see Fig. 3).

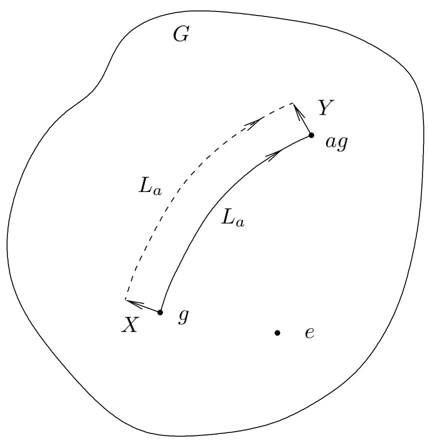

Fig. 3. A Lie group G is a differentiable manifold with a privileged point e (the identity). Since $L_a : g \mapsto ag$, $L_{a*}|_g$ maps T_gG to $T_{ag}G$. Intuitively, the tangent map proceeds by mapping both the base and tip of an infinitesimal displacement at g (the vector X in the figure) by the map L_a, to produce an infinitesimal displacement at ag (the vector Y in the figure).

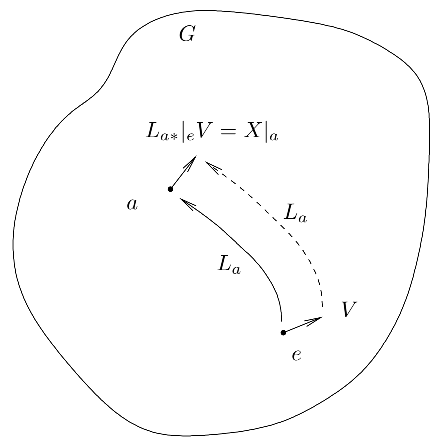

Fig. 4. By left-translating a vector V at the identity, we can create a vector field on G. This vector field is left-invariant.

The space of LIVF's is a subset of $\mathfrak{X}(G)$, a rather small subset, in fact, as we see when we note from Eq. (14) that a LIVF is determined at all points of G once its value is known at one point of G. In fact, the identity is a convenient reference location. Let $X \in \mathfrak{X}(G)$ be a LIVF, and let $V = X|_e$. See Fig. 4. Note that $V \in T_eG$ is a vector at a point, not a vector field. Then by setting $g = e$ in Eq. (14), we have

$$X|_a = (L_{a*}|_e)V, \qquad \forall a \in G. \tag{15}$$

Thus, every LIVF can be associated with a vector in T_eG (its value at e). Conversely, let V be any vector in T_eG, and define a vector field $X \in \mathfrak{X}(G)$ by Eq. (15). Then this vector field is left-invariant, as we see by left-translating it:

$$(L_{b*}|_a)(X|_a) = (L_{b*}|_a)(L_{a*}|_e)V = (L_b L_a)|_{*e}V = (L_{ba*})|_e V = X|_{ba}, \tag{16}$$

where we use the property of the tangent map, $F_* G_* = (FG)_*$, and the fact that $a \mapsto L_a$ is an action. Thus, every vector $V \in T_e G$ is associated with a LIVF. We see that there is a one-to-one correspondence between LIVF's and vectors in $T_e G$, given by Eq. (15).

Notice that the association is linear. Since $T_e G$ is a vector space isomorphic to $\mathbb{R}^n$ (where $n = \dim G$), so is the space of LIVF's.

Henceforth we will write X_V, X_W, etc., to denote the LIVF's whose value at e is V, W, etc. To go from V to X_V, we use Eq. (15); to go from X_V to V we just evaluate at the identity,

$$V = X_V|_e. \tag{17}$$

10. The Lie Algebra

The space $T_e G$ is called the *Lie algebra* of the group G. It is denoted $\mathfrak{g}$, and it is a real, n-dimensional vector space, isomorphic to the space of LIVF's, where $n = \dim G$. Why it is called a Lie algebra will be explained momentarily.

Note that since L_a is a diffeomorphism, $L_{a*}|_e : \mathfrak{g} \to T_a G$ has full rank, so Eq. (15) can be used to map a basis $\{V_\mu, \mu = 1, \ldots, n\}$ in $\mathfrak{g}$ into a basis in $T_a G$. Said another way, the set of LIVF's defined by

$$X_\mu = X_{V_\mu} \quad \text{or} \quad X_\mu|_a = L_{a*}|_e V_\mu \tag{18}$$

are linearly independent at each point $a \in G$ (they form a basis of LIVF's). In a homework problem it is shown that a basis of vector fields $\{e_\mu\}$ (generally only defined locally) is a coordinate basis, that is, $e_\mu = \partial/\partial x^\mu$ for some coordinates x^μ, if and only if $[e_\mu, e_\nu] = 0$. As we will see, the LIVF's $\{X_\mu\}$ generally do not commute and thus are not a coordinate basis. They are, however, a particularly useful basis when tensors must be expressed in components.

Note also that the basis vector fields $\{X_\mu\}$ are defined everywhere on G (not just locally). We see that it is always possible to define a smooth set of frames in the tangent spaces over all of G, for any Lie group (the frames are the linearly independent values of the fields X_μ at points of G). This cannot be done on just any manifold. For example, on the sphere S^2 there does not exist a pair of smooth vector fields that are linearly independent at every point. In fact, there does not exist even one vector field that is linearly independent at each point, that is, that vanishes nowhere. This is the "hair on the coconut" theorem. Accepting this theorem, we see that S^2 cannot be a group manifold. On the other hand, S^3 is a group manifold, that of $SU(2)$, and possesses a global frame.

The set of LIVF's on G is closed under the Lie bracket. This follows easily from the rule, $F_*[X, Y] = [F_* X, F_* Y]$, valid when F is a diffeomorphism. In the present case, let X_V and X_W be two LIVF's associated with $V, W \in \mathfrak{g}$. Then

$$L_{a*}[X_V, X_W] = [L_{a*} X_V, L_{a*} X_W] = [X_V, X_W], \tag{19}$$

where we use Eq. (13a) in the last step. Thus, the Lie bracket of two LIVF's is a LIVF. This new LIVF must be the left translate of some vector U at the identity, that is, writing $[X_V, X_W]|_e = U$, we must have $[X_V, X_W] = X_U$. We now write

$$U = [V, W], \tag{20}$$

thereby defining the bracket operation $[\,,\,]$ on $\mathfrak{g}$. In words, to compute $[V, W]$ for $V, W \in \mathfrak{g}$, we first promote V and W into vector fields X_V, X_W by left translation, when then compute the Lie bracket of these vector fields, then we evaluate the resulting vector field at e. This is not the Lie bracket, which is not defined on vectors at a point (only on vector fields), but rather a new bracket operation. Note that on an arbitrary manifold (not a group), there is no meaning to the bracket of two vectors in a single tangent space. It is only because of the group structure that this is meaningful on a Lie group. We can summarize the above relations by writing,

$$[X_V, X_W]|_e = [V, W], \tag{21}$$

or, by left translating both sides,

$$[X_V, X_W] = X_{[V,W]}. \tag{22}$$

The new bracket operation on $\mathfrak{g}$ is antisymmetric and satisfies the Jacobi identity. The latter follows from the Jacobi identity for vector fields, since if $U, V, W \in \mathfrak{g}$, then

$$[U, [V, W]] = [X_U, X_{[V,W]}]|_e = [X_U, [X_V, X_W]]|_e, \tag{23}$$

where we use Eqs. (21) and (22). If we cycle U, V, and W in this expression and use the Jacobi identity on the Lie bracket for vector fields, we find that vectors in $\mathfrak{g}$ also satisfy the Jacobi identity. Thus, $\mathfrak{g}$ is a Lie algebra in the technical sense of that phrase.

[A *Lie algebra* is a real vector space A with a bracket operation $[\,,\,] : A \times A \to A$, such that the bracket is linear in both operands, antisymmetric, and satisfies the Jacobi identity.]

The space of vector fields $\mathfrak{X}(M)$ on any manifold M is a Lie algebra under the Lie bracket, $[\,,\,] : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$. Equation (21) defines the bracket operation on the Lie algebra of the group, $[\,,\,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, and the process of left translation defines a map : $\mathfrak{g} \to \mathfrak{X}(G)$. By Eq. (22), this map is a Lie algebra homomormphism.

11. Advance Maps of Left-Invariant Vector Fields

Let us now consider the advance maps and integral curves associated with LIVF's. Let X_V be the LIVF associated with $V \in \mathfrak{g}$, and let $\Phi_{V,t}$ be the advance map. Let $\sigma : \mathbb{R} \to G$ be the integral curve passing through e at $t = 0$, that is, let

$$\sigma(t) = \Phi_{V,t}\, e. \tag{24}$$

See Fig. 5. Then it turns out that other integral curves of X_V passing through other points at $t = 0$ can be expressed in terms of $\sigma(t)$.

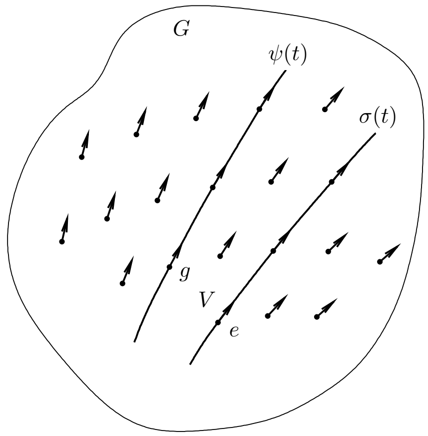

Fig. 5. A left-invariant vector field X is illustrated, corresponding to $V \in \mathfrak{g}$. The integral curve passing through e at $t = 0$ is $\sigma(t)$, and the one passing through g at $t = 0$ is $\psi(t)$.

To prove this we use the following fact. If $F : M \to N$ is a diffeomorphism between manifolds M and N, and $X \in \mathfrak{X}(M)$, so that $F_* X \in \mathfrak{X}(N)$, and if Φ_t is the advance map for X and Ψ_t is the advance map for $F_* X$, then

$$F\Phi_t\, x = \Psi_t F\, x, \qquad \forall x \in M. \tag{25}$$

This fact was proved in a homework exercise (it is simply the chain rule plus the uniqueness theorem when expressed in coordinates). It can also be written,

$$Fe^{tX} = e^{tF_* X} F, \tag{26}$$

using the formal exponental notation for advance maps.

In the present case we identify both M and N with G, we identify F with L_a for some $a \in G$, and we identify X with X_V, a LIVF. Then Eq. (26) becomes

$$L_a \Phi_{V,t} = L_a e^{tX_V} = e^{tL_{a*} X_V}\, L_a = e^{tX_V} L_a = \Phi_{V,t} L_a, \tag{27}$$

where we use $L_{a*} X_V = X_V$. We see that the advance maps for the flow of a left-invariant vector field commute with left translations. Now let $\psi(t) = \Phi_{V,t}\, g$ be the integral curve of X_V passing through g at $t = 0$. Then we have

$$\psi(t) = \Phi_{V,t}\, g = \Phi_{V,t} L_g\, e = L_g \Phi_{V,t}\, e = L_g\, \sigma(t). \tag{28}$$

We may abbreviate this by writing,

$$\Phi_{V,t}\, g = g\sigma(t) = R_{\sigma(t)}\, g, \tag{29}$$

or simply,

$$\Phi_{V,t} = R_{\sigma(t)}. \tag{30}$$

As claimed, an arbitrary integral curve of X_V can be expressed in terms of the special integral curve $\sigma(t)$ passing through e at $t = 0$.

Now suppose g lies on σ, that is, let $g = \sigma(s)$ for some s. Then

$$\Phi_{V,t}\, g = \Phi_{V,t}\, \sigma(s) = \Phi_{V,t}\Phi_{V,s}\, e = \Phi_{V,s+t}\, e = \sigma(s+t) = g\sigma(t) = \sigma(s)\sigma(t), \tag{31}$$

where we use Eq. (29) and the composition rule for advance maps. We summarize this by writing,

$$\sigma(s)\sigma(t) = \sigma(s+t) = \sigma(t)\sigma(s). \tag{32}$$

Thus the integral curve $\sigma : \mathbb{R} \to G$ of X_V passing through e at $t = 0$ is a group homomorphism (where $\mathbb{R}$ is a group under addition). Such a homomorphism of $\mathbb{R}$ onto a group G is called a *one-parameter subgroup*. We have shown that every LIVF (hence every element of $\mathfrak{g}$) corresponds to a one-parameter subgroup.

Conversely, any one-parameter subgroup $\sigma : \mathbb{R} \to G$ has a tangent vector $V = \sigma'(0) \in \mathfrak{g}$ at $t = 0$, associated with a LIVF X_V of which σ is an integral curve. Altogether, we see that there is a one-to-one association between vectors in the Lie algebra $\mathfrak{g}$, left-invariant vector fields, and one-parameter subgroups.

$$SO(3) = \frac{D^3}{\sim \text{(antipodal)}}$$

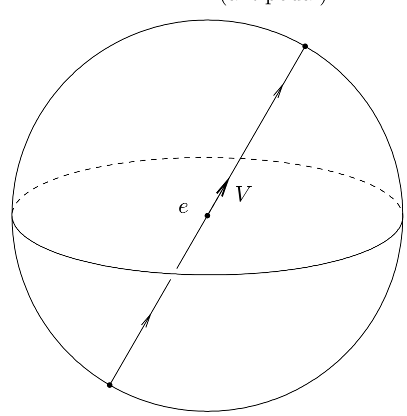

Fig. 6. The group $SO(3)$ is represented as the 3-disk D^3 with antipodal points on the surface identified. The identity element is in the center, and a 1-parameter subgroup is a straight line that proceeds from the center to the surface, continuing from the antipodal point back to the center again. The one-parameter subgroup consists of rotations about a fixed axis, specified by V; the subgroup itself is $SO(2)$ (a circle).

It is easy to visualize the one-parameter subgroups in the case of $SO(3)$. We use the 3-disk model of $SO(3)$, in which $SO(3)$ is D^3 of radius π in $\boldsymbol{\theta} = (\theta_x, \theta_y, \theta_z)$-space with antipodal points on the surface S^2 identified (see Fig. 6). Then the identity element is at the center and the one-parameter subgroups are straight lines passing through the identity, progressing to the surface whereupon they reappear at the antipodal point, and continuing on a straight line until they return to the identity. Topologically, these subgroups are circles. The vector $V \in \mathfrak{g}$ indicates the initial direction of the line, and its magnitude indicates the rate at which the one-parameter subgroup is traversed. Geometrically, the subgroup is the $SO(2)$ subgroup of rotations about the axis indicated by V. The picture with $SU(2)$ is similar, except the sphere has radius 2π and all points on the S^2 surface of the sphere are identified as a single point (the south pole of S^3, if the identity is placed at the north pole).

12. The Exponential Map

If a vector $V \in \mathfrak{g}$ is scaled by some constant factor, $V \mapsto kV$ for $k \in \mathbb{R}$, then $X_V \mapsto kX_V$. This does not change the integral curves regarded as subsets of G, but it does change their parameterization, causing them to be traversed k times as fast. Therefore to traverse the same amount of an integral curve, we should scale the time by $t \mapsto t/k$. Thus we have the identity,

$$\Phi_{V,t} = \Phi_{kV,t/k} = \Phi_{tV,1}, \tag{33}$$

where in the last equality we have set $t = k$. The advance map $\Phi_{V,t}$ actually depends only on the product tV.

This leads to a definition of the *exponential map* $\exp : \mathfrak{g} \to G$, given by

$$\exp(V) = \Phi_{V,1}\, e, \tag{34}$$

or,

$$\exp(tV) = \Phi_{V,t}\, e. \tag{35}$$

This is one of several uses of the symbol exp in differential geometry, and in this case it is not to be interpreted literally as a power series (but see below regarding matrix groups).

The exponential map can be thought of intuitively as follows. Let $V \in \mathfrak{g}$. Multiplying this by a small scale factor ϵ, the vector ϵV can be thought of as a small displacement at the identity, taking us from e to point "$e + \epsilon V$". We put this expression in quotes because in general there is no way to add points of a manifold, but intuitively we are identifying a small region of G in the neighborhood of the identity with the tangent space, which is a vector space. This small displacement can be thought of as a small step in following the left-invariant vector field X_V, starting at e. To take successive steps, we just compute powers (square, cube, etc) of the small group element $e + \epsilon V$. There is a limit of elementary calculus,

$$\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x, \tag{36}$$

where x is a number. This limit suggests the notation

$$\lim_{n\to\infty} \left(e + \frac{V}{n}\right)^n = \exp(V), \tag{37}$$

where ϵ is identified with $1/n$, which conveys the correct idea of the exponential map, as well as giving an intuitive idea of why the integral curve of the left-invariant vector field through e is actually a one-parameter subgroup. For example, in Fig. 6, the one-parameter subgroup of $SO(3)$ is generated by compounding small rotations about a fixed axis (the direction of V in $\boldsymbol{\theta}$-space).

A question is whether any point of the group can be reached by exponentiating some element of the Lie algebra, that is, whether the exponential map is onto (surjective). In fact, exp is onto for connected, compact Lie groups, but not generally otherwise.

Whether or not exp is onto, the exponential map provides a coordinate system on the group manifold in some neighborhood of the identity. We simply choose some coordinates in $\mathfrak{g}$ (by choosing a basis $\{V_\mu\}$), and then use the exponential map to identify coordinates in $\mathfrak{g}$ with coordinates on G itself. We will call these *exponential coordinates*. Such coordinates have much in common with Riemann normal coordinates in Riemannian geometry, a topic we will consider later. In the case of $SO(3)$, the $\boldsymbol{\theta} = (\theta_x, \theta_y, \theta_z)$ coordinates discussed earlier in class in the 3-disk model of $SO(3)$ are exponential coordinates. If you have to use coordinates on a group manifold (something to be avoided if possible), exponential coordinates may be the best choice.

13. Left-Invariant Forms

Just as we created left-invariant vector fields by left-translating a vector at the identity, we can create left-invariant 1-forms by left-translating a covector at the identity. More generally, a differential r-form at the identity can be left-translated to create a left-invariant field of differential r-forms on G (an element of $\Omega^r(G)$).

We begin with some notation regarding pull-backs. See Sec. 7 for the corresponding notation regarding tangent maps (also called "push-forwards"). If $A : V \to W$ is a linear map between real vector spaces, then the pull-back of A is a linear map $A^* : W^* \to V^*$, defined by $A^*\beta = \beta \circ A$, where both sides act on V (they are both forms in V^*). Now identify A with the tangent map $F_*|_x$, as in Figs. 1 and 2, where $F_*|_x : T_xM \to T_{F(x)}N$. Then the pull-back of this map is $(F_*|_x)^*$, for which we write simply $F^*|_x$. It is a map,

$$F^*|_x : T^*_{F(x)}N \to T^*_xN. \tag{38}$$

If β is a form in $T^*_{F(x)}N$ and X is a vector in T_xM, then

$$(F^*|_x\,\beta)(X) = \beta(F_*|_x\,X). \tag{39}$$

It may seem illogical to label the pull-back $F^*|_x$ by x, the point at which the target space (the range of the map) is attached, while in the case of tangent maps the x in $F_*|_x$ was a label of the domain of the map. However, since in general F^{-1} does not exist, these maps cannot be labelled by $F(x)$, since more than one $x \in M$ might correspond to the same $F(x) \in N$ (the map F need not be injective). We are labelling both the tangent map $F_*|_x$ and the pull-back $F^*|_x$ by the same label (the point $x \in M$).

Equation (38) applies to a single form β attached to $F(x) \in N$, mapping it to a single form attached to $x \in M$. If we reinterpret β as a field of 1-forms, that is, a member of $\mathfrak{X}^*(N) = \Omega^1(N)$, and X as a vector field on M, then Eq. (39) becomes

$$(F^*|_x\,\beta|_{F(x)})(X|_x) = \beta|_{F(x)}(F_*|_x\,X|_x). \tag{40}$$

This specifies a covector field on M, denoted $F^*\beta$ (without the "$|_x$" attached). The relation between the notations is

$$(F^*\beta)|_x = (F^*|_x)(\beta|_{F(x)}). \tag{41}$$

Here F^* (the "pull-back" of F) is regarded as a map,

$$F^* : \Omega^1(N) \to \Omega^1(M). \tag{42}$$

Unlike the case of the push-forward of vector fields, the pull-back F^* in this sense is defined for all smooth maps $F : M \to N$ (in Eq. (11) F had to be a diffeomorphism).

Now return to Lie groups. Let the space dual to $\mathfrak{g}$ be $\mathfrak{g}^*$. It consists of linear maps : $\mathfrak{g} \to \mathbb{R}$. Let $\beta \in \mathfrak{g}^*$ be a single 1-form at the identity. To left-translate this to another point $a \in G$, we must have a map from $\mathfrak{g}^* = T^*_e G$ to $T^*_a G$. This is not the pull-back of L_a, which maps $T^*_a G$ to $T^*_e G$ (it goes in the wrong direction). Instead we must use the pull-back of $L_{a^{-1}}$. Therefore we define a 1-form $\theta \in \Omega^1(G)$ by

$$\theta|_a = L^*_{a^{-1}}|_a\, \beta. \tag{43}$$

.

This form is left-invariant, that is,

$$L^*_g \theta = \theta, \qquad \forall g \in G, \tag{44}$$

where we use the pull-back in the sense of Eq. (42) (with $M = N = G$). To prove this, we evaluate the left-hand side of Eq. (44) at $a \in G$, and use Eq. (41) and (43),

$$(L^*_g\theta)|_a = (L^*_g|_a)(\theta|_{ga}) = L^*_g|_a\, L^*_{(ga)^{-1}}|_{ga}\, \beta. \tag{45}$$

The product of pull-back maps in this equation is defined, since

$$\begin{aligned} L^*_{(ga)^{-1}}|_{ga} &: T^*_e G \to T^*_{ga} G, \\ L^*_g|_a &: T^*_{ga} G \to T^*_a G, \end{aligned} \tag{46}$$

and in fact

$$L^*_g|_a\, L^*_{(ga)^{-1}}|_{ga} = (L_{(ga)^{-1}} L_g)^*|_a = L^*_{a^{-1}}|_a, \tag{47}$$

where we follow the rule that the order of the composition of maps is reversed on taking the pull-back. Therefore the expression (45) becomes

$$L^*_{a^{-1}}|_a\, \beta = \theta|_a, \tag{48}$$

which proves Eq. (44).

In this way we define left-invariant 1-forms on G. If θ is a left-invariant 1-form defined by left-translating $\beta \in \mathfrak{g}^*$, as in Eq. (43), and X is a left-invariant vector field that is obtained by left-translating $V \in \mathfrak{g}$, as in Eq.(15) (for simplicity we write X instead of X_V), then we can evaluate θ on X at some group element g. This is

$$\theta|_g(X|_g) = (L^*_{g^{-1}}|_g\, \beta)(L_{g*}|_e\, V) = \beta(L_{g^{-1}*}|_g\, L_{g*}|_e\, V) = \beta((L_{g^{-1}} L_g)_{*e}\, V) = \beta(V). \tag{49}$$

We see that $\theta(X)$ is the same at g as it is at e (and therefore it has the same value everywhere on G). This makes sense: By pairing a left-invariant 1-form with a left-invariant vector field we get a left-invariant scalar field. But a left-invariant scalar must have the same value everywhere, that is, it must be a constant.

This has an implication for basis vectors and forms. Let $\{X_\mu, \mu = 1, \ldots, \dim G\}$ be a set of left-invariant vector fields associated with a basis $\{V_\mu\}$ in $\mathfrak{g}$, as defined in Eq. (18). As explained previously, the vector fields X_μ are linearly independent everywhere on G, defining a frame in each tangent space. Now let $\{\beta^\mu\}$ be a set of 1-forms in $\mathfrak{g}^*$ that is dual to the basis $\{V_\mu\}$ in $\mathfrak{g}$, that is, let

$$\beta^\mu(V_\nu) = \delta^\mu_\nu. \tag{50}$$

Then if we left translate the basis $\{\beta^\mu\}$ to create a set of (fields of) 1-forms $\{\theta^\mu\}$, where $\theta^\mu \in \Omega^1(G)$ for each $\mu = 1, \ldots, \dim G$, then their action on the basis vector fields $\{X_\mu\}$ have the same values everywhere as at the identity, that is, δ^μ_ν as in Eq. (50). In other words, the basis of 1-forms $\{\theta^\mu\}$ is dual to the basis of vector fields $\{X_\mu\}$ at every point of G.

14. Fields of Frames

This is an opportunity to speak of fields of frames in general, that is, on any manifold. We have just seen that fields of frames arise naturally on group manifolds, but they are useful in many other contexts, too. Let M be a manifold, let $n = \dim M$ and let $\{e_\mu, \mu = 1, \ldots, n\}$ be a set of vector fields that are linearly independent at each point over some region of M. Although group manifolds always possess global frames, this is not true of other manifolds, so in general the constructions we are about to present are only local. For this reason it may be best to think of the vector fields e_μ as defined only over some open set $U \subset M$.

Certainly if we have a coordinate chart defined over some open set $U \subset M$, then we obtain a field of frames over this set by letting

$$e_\mu = \frac{\partial}{\partial x^\mu}. \tag{51}$$

Given a field of frames $\{e_\mu\}$, we will call it a *coordinate frame* if there exist local scalar fields x^μ such that Eq. (51) holds. Obviously, in the case of a coordinate frame, we have

$$[e_\mu, e_\nu] = 0, \tag{52}$$

since the partial derivative operators (51) commute with one another, so if we have a frame for which the condition (52) does not hold then the frame is not a coordinate frame. In particular, this condition does not hold for left-invariant frames on the manifold G of a non-Abelian group. In a homework problem it was asked to show that a field of frames $\{e_\mu\}$ is a coordinate basis if and only if Eq. (52) is satisfied. The hard part is to show that if Eq. (52) is satisfied, then coordinates x^μ exist such that Eq. (51) is satisfied. In the homework problem it was expected that you would just work this out in coordinates. A more elegant proof will be given below.

Given a field of frames, the Lie brackets of the vector fields among themselves, whatever they are, are themselves vector fields, so they can expressed as linear combinations of the basis fields. That is, there exist coefficients $c^\sigma_{\mu\nu}$ such that

$$[e_\mu, e_\nu] = c^\sigma_{\mu\nu}\, e_\sigma. \tag{53}$$

These coefficients are called *structure constants*, a bad terminology since they are not constant (they depend on position on the manifold, in general). By whatever name, however, these coefficients are useful. Because of the antisymmetry of the Lie bracket, the structure constants are antisymmetric in their lower two indices,

$$c^\sigma_{\mu\nu} = -c^\sigma_{\nu\mu}. \tag{54}$$

Given a field of frames $\{e_\mu\}$ and the dual basis of 1-forms $\{\theta^\mu\}$, we can compute the components of tensors with respect to these in the usual way. For example, if g is a metric tensor, its components with respect to the basis $\{e_\mu\}$ are

$$g_{\mu\nu} = g(e_\mu, e_\nu), \tag{55}$$

and the components of a vector field Y are

$$Y^\mu = Y(\theta^\mu) = \theta^\mu(Y). \tag{56}$$

Sometimes in the literature people distinguish the components of tensors with respect to a coordinate basis from those with respect to a non-coordinate basis by using different indices (for example, Greek for one kind and Roman for the other). In this course, however, we will not follow any rigid rules about this. Rather when writing tensors and tensor-like objects in components we will henceforth take a non-coordinate basis as a default, and make an explicit note if we intend a coordinate basis.

It is of interest to compute the exterior derivatives of the basis forms θ^μ. The components of these forms can be computed by the general formula for the exterior derivative of a form acting on a set of vector fields, which in this case (the exterior derivative of a 1-form) gives

$$d\theta^\mu(e_\sigma, e_\tau) = e_\sigma \theta^\mu(e_\tau) - e_\tau \theta^\mu(e_\sigma) - \theta^\mu([e_\sigma, e_\tau]), \tag{57}$$

where the first term on the right means the action of the vector field e_σ (a differential operator) on the scalar $\theta^\mu(e_\tau)$, and similarly for the second term. But these two terms vanish, since $\theta^\mu(e_\tau) = \delta^\mu_\tau$ and $\theta^\mu(e_\sigma) = \delta^\mu_\sigma$, which are constants. As for the third term, the Lie bracket can be expanded according to Eq. (53). Thus we find the components of $d\theta^\mu$,

$$d\theta^\mu(e_\sigma, e_\tau) = -c^\mu_{\sigma\tau}. \tag{58}$$

Then expressing the 2-form $d\theta^\mu$ in terms of its components, we have

$$d\theta^\mu = -\frac{1}{2} c^\mu_{\sigma\tau}\, \theta^\sigma \wedge \theta^\tau. \tag{59}$$

This is a differential equation satisfied by the basis of 1-forms, analogous to Eq. (53), which is satisfied by the basis of vector fields. Equation (59) is essentially the same as the Maurer-Cartan equation, which is discussed by Nakahara.

Now suppose the basis $\{e_\mu\}$ is a coordinate basis, so $e_\mu = \partial/\partial x^\mu$, for some local scalar fields x^μ. Then the dual basis is $\theta^\mu = dx^\mu$, and $d\theta^\mu = ddx^\mu = 0$, since $dd = 0$. Then Eq. (59) implies $c^\mu_{\sigma\tau} = 0$, which in turn implies $[e_\mu, e_\nu] = 0$. We knew this already, but it is interesting to see the same conclusions emerge from a treatment of the dual basis. As for the converse, suppose $[e_\mu, e_\nu] = 0$, so $c^\mu_{\sigma\tau} = 0$. This implies $d\theta^\mu = 0$, that is, the 1-forms of the dual basis are closed. Now the question is, are they also exact, that is, do there exist 0-forms (scalars) x^μ (which only need to be defined locally) such that $\theta^\mu = dx^\mu$? The answer is yes, locally a closed form is always exact. This is called the *Poincaré lemma*, and we will discuss it in more detail later. Accepting this, we see that if the basis vector fields commute, then there exist local scalars x^μ such that $\theta^\mu = dx^\mu$, and the basis is a coordinate basis.

Another perspective on this question is the following. There is a theorem of differential geometry that if two vector fields commute, then their advance maps commute. As we have explained, the Lie bracket of two vector fields is related to the first nonvanishing term in the Taylor series of the commutator of the two advance maps, properly defined. So if the Lie bracket vanishes, then this commutator also vanishes to second order. But it is a fact that if it vanishes to second order, then it vanishes exactly. Now suppose the vector fields e_μ commute, so that their advance maps also commute. To start from a given point of $U \subset M$ and to reach another point, we can apply these advance maps with certain unique elapsed parameters, in any order, to the initial point. Call the elapsed parameters x^μ, with $x^\mu = 0$ at the initial point. Then these are the coordinates for which $e_\mu = \partial/\partial x^\mu$.

15. Frames on Lie Groups

To return to the case of Lie groups, let $\{V_\mu\}$ be a basis in $\mathfrak{g}$, as above. Then the Lie algebra bracket of two of the basis vectors is another vector in $\mathfrak{g}$, which can be expanded as a linear combination of the basis vectors. That is, we can write,

$$[V_\mu, V_\nu] = \bar{c}^\sigma_{\mu\nu}\, V_\sigma, \tag{60}$$

where we write $\bar{c}^\sigma_{\mu\nu}$ with an overbar to indicate that these are the structure constants for the bracket $[\,,\,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, not the Lie bracket of vector fields $[\,,\,] : \mathfrak{X}(G) \times \mathfrak{X}(G) \to \mathfrak{X}(G)$, as in Eq. (53). The coefficients $\bar{c}^\sigma_{\mu\nu}$ in this context are just numbers, not scalar fields on the manifold as in Eq. (53).

But if we left-translate both sides of Eq. (60), writing e_μ for the left-invariant vector field that is the left-translate of V_μ, then we have

$$[e_\mu, e_\nu] = \bar{c}^\sigma_{\mu\nu}\, e_\sigma, \tag{61}$$

showing that the structure constants for the field of left-invariant frames $\{e_\mu\}$ are the same as the structure constants of the Lie algebra, and, in fact, they are independent of position on G. That is, we can drop the overbar in Eq. (60).

In this context it is more appropriate to call the $c^\sigma_{\mu\nu}$ "structure constants," since they do not depend on $g \in G$. They still depend, however, on the choice of basis $\{V_\mu\}$ in $\mathfrak{g}$. They may be thought of as the components of a type $(1, 2)$ tensor on $\mathfrak{g}$. The structure constants are usually thought of as a characteristic of the group (but we should keep in mind that they also depend on the choice of basis in $\mathfrak{g}$). Intuitively, they give the multiplication law for group elements in an infinitesimal neighborhood of the identity.

A question is whether two groups whose multiplication laws agree in a small neighborhood of the identity (that is, two groups whose Lie algebras are isomorphic) are themselves isomorphic. The answer is no, two groups with the same Lie algebra can be placed in one-to-one correspondence of some (finite) neighborhood of the identity, but need not be isomorphic globally. Generally speaking, one group is a covering space of the other, and there exists a projection map from the covering group to the covered group, which is a group homomorphism. Examples of this include $SU(2)$ which covers $SO(3)$ twice, and $\mathbb{R}$ which covers $U(1)$ an infinite number of times.

16. The Adjoint Representation

Let us return to the inner automorphism I_g, defined in Eq. (8). The map $g \to I_g$ is an action of G on itself, as noted in Sec. 6. The orbits of this action are called the *conjugacy classes* of the group; they consist of sets of group elements that are conjugate to one another ($a, b \in G$ are conjugate if there exists $g \in G$ such that $a = gbg^{-1}$). The quotient space of this action (G divided by the equivalence relation induced by the action $g \to I_g$) is the space of conjugacy classes.

As an aside, we will mention the role played by this quotient space in representation theory, which is important in many physical applications. The space of conjugacy classes is in a sense the space conjugate to the space of irreducible representations. That is, there exists a set of "wave functions" on this space that are orthonormal with respect to a natural measure on this space. The "quantum numbers" of these wave functions are the labels of the irreducible representations of the group. These "wave functions" are called the "characters" of the representation.

The identity element e is always in a conjugacy class by itself, since $geg^{-1} = e$ for all $g \in G$. Thus,

$$I_g e = e, \qquad \forall g \in G. \tag{62}$$

This means that the tangent map of I_g, evaluated at the identity, is a linear map of the Lie algebra into itself,

$$I_{g*}|_e : T_e G \to T_e G. \tag{63}$$

This map actually constitutes an action of G on the Lie algebra, called the *adjoint action*, and it has a special notation,

$$\mathrm{Ad}_g = I_{g*}|_e, \qquad \mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}. \tag{64}$$

To show that $g \to \mathrm{Ad}_g$ is an action, we write

$$\mathrm{Ad}_g\, \mathrm{Ad}_h = I_{g*}|_e\, I_{h*}|_e = R_{g^{-1}*}|_g\, L_{g*}|_e\, R_{h^{-1}*}|_h\, L_{h*}|_e, \tag{65}$$

using Eq. (9). Now working on the middle two factors we have

$$L_{g*}|_e\, R_{h^{-1}*}|_h = (L_g\, R_{h^{-1}})_*|_h = (R_{h^{-1}}\, L_g)_*|_h = R_{h^{-1}*}|_{gh}\, L_{g*}|_h, \tag{66}$$

where we use Eq. (9) again. Now combining the first factor on the right hand side of Eq. (65) with the first of Eq. (66), we have

$$R_{g^{-1}*}|_g\, R_{h^{-1}*}|_{gh} = (R_{g^{-1}}\, R_{h^{-1}})_*|_{gh} = R_{(gh)^{-1}*}|_{gh}. \tag{67}$$

Similarly treating the last factors in Eqs. (66) and (65) we have

$$L_{g*}|_h\, L_{h*}|_e = (L_g\, L_h)_*|_e = L_{gh*}|_e, \tag{68}$$

and thus

$$\mathrm{Ad}_g\, \mathrm{Ad}_h = R_{(gh)^{-1}*}|_{gh}\, L_{gh*}|_e = (R_{(gh)^{-1}}\, L_{gh})_*|_e = \mathrm{Ad}_{gh}. \tag{69}$$

An action of a group on a vector space by linear operators is called a *representation*, and the adjoint representation $g \mapsto \mathrm{Ad}_g$ is a representation that every group carries on its own Lie algebra. It plays an important role in the general theory of representations.