manifold C.

Whenever an algebraic equation like (12b) emerges as one of the Euler-Lagrange equations, there is a temptation to use it to eliminate some of the variables in the Lagrangian, to obtain a new variational principle with a smaller number of variables. Then the question arises, is the new (reduced) variational principle equivalent to the original one, that is, does it give an equivalent system of equations.

The answer is no, not in general, when we do this in general we find that the solution set enlarges. If we regard the original variational principle as the "physical" one, then sometimes when we eliminate variables in this way we obtain "nonphysical" (extra) solutions.

It would take a while to explain this in detail, but in any specific example it is easy just to check if the reduced system is equivalent to the original one.

For example, let us take the constraint (12b), which is $p = \partial L/\partial v$, giving p as a function of q, v, t. Obviously we can use this to eliminate from the action (9); if we do so, we find a reduced action,

$$\tilde{a}[q(t), v(t)] = \int_{t_0}^{t_1} L(q, v, t)\,dt + \frac{\partial L}{\partial v}(\dot{q} - v). \qquad (30)$$

If we work out the E-L equations from this, we find that they are equivalent to (12a) and (12c) (with (12b) understood) if the Lagrangian L is regular, i.e. if

$$\frac{\partial^2 L}{\partial v^2} \neq 0. \qquad (31).$$

On the other hand, if $\partial^2 L/\partial v^2 \neq 0$ (a regular Lagrangian), then $p = p(q, v, t)$ can be solved to get v as a function of (q, p, t). Doing this and substituting into (9), we get

$$\bar{a}[q(t), p(t)] = \int_{t_0}^{t_1} (p\dot{q} - H)\, dt, \qquad (32)$$

~~which~~ for which the E-L equations are the usual Hamilton's equations,

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q} \qquad (33).$$

In (32) H is now the usual Hamiltonian $H(q, p, t)$. It is the same as (11), except v has been expressed as a function of (q, p, t). The eqns (33) are equivalent to (12) if the Lagrangian is regular.

The geometrical meaning of taking the constraint equation (12b) and using it to eliminate variables is that we are taking the θ in (18), $\theta \in \Omega^1$(qvpt-space) and replacing it by $\theta|_C \in \Omega^1(C)$. If we eliminate p

using $p = \partial L/\partial v$ it is equivalent to using coordinates (q, v, t) on C. That is, C is diffeomorphic to (q, v, t)-space, which otherwise is the time-extended tangent bundle, $TM \times \mathbb{R}$.

$$\theta|_C = L(q, v, t)\,dt + \frac{\partial L}{\partial v}\,(dq - v\,dt). \tag{34}$$

Then the new variational principle is

$$\delta \int_\gamma \theta|_C = 0 \tag{35}$$

where now γ is understood to be confined to C. And, as mentioned, this is equivalent to the original one if the Lagrangian is regular.

If X is tangent to γ (now restricted to C), where γ is a solution of (35), then

$$i_X\, d(\theta|_C) = 0, \tag{36}$$

by the mathematical logic as before. But restricting a form is the same as pulling it back by the inclusion map, so $d(\theta|_C) = (d\theta)|_C = \omega|_C$. Thus the new version of the Euler-Lagrange equations are

$$i_X\, \omega|_C = 0 \tag{37}$$

Similarly, if the Lagrangian is regular, then we can use coordinates (q,p,t) on C, and

$$C \cong T^*M \times \mathbb{R}, \tag{38}$$

the time-extended cotangent bundle. Then in terms of these coordinates,

$$\theta|_C = p\,dq - H(q,p,t)\,dt. \tag{39}$$

For regular Lagrangians, this is the same as (34), just expressed in different coordinates.

Since C is 3-dimensional, the rank of $\omega_{\mu\nu}$ can only be 0 or 2. In fact, in the coordinates (q,p,t), we have

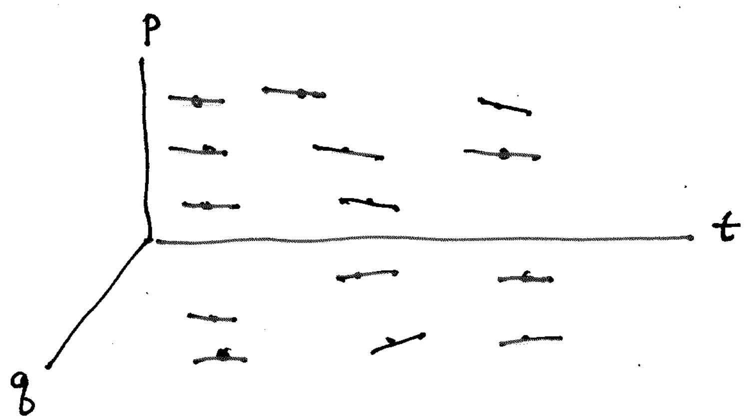
$$(\omega|_C)_{\mu\nu} = \begin{array}{c} \\ q \\ p \\ t \end{array}\!\!\begin{array}{c} \begin{array}{ccc} q & p & t \end{array} \\ \begin{pmatrix} 0 & -1 & -\partial H/\partial q \\ 1 & 0 & -\partial H/\partial p \\ \dfrac{\partial H}{\partial q} & \dfrac{\partial H}{\partial p} & 0 \end{pmatrix} \end{array} \tag{40}$$

In this case, $\operatorname{rank}(\omega|_C)_{\mu\nu} = 2$, so $\dim \ker \omega_{\mu\nu} = 1$ everywhere on $C = T^*M \times \mathbb{R}$. From this it follows that if X is tangent to a solution curve $\gamma \subset C$, then X is proportional to

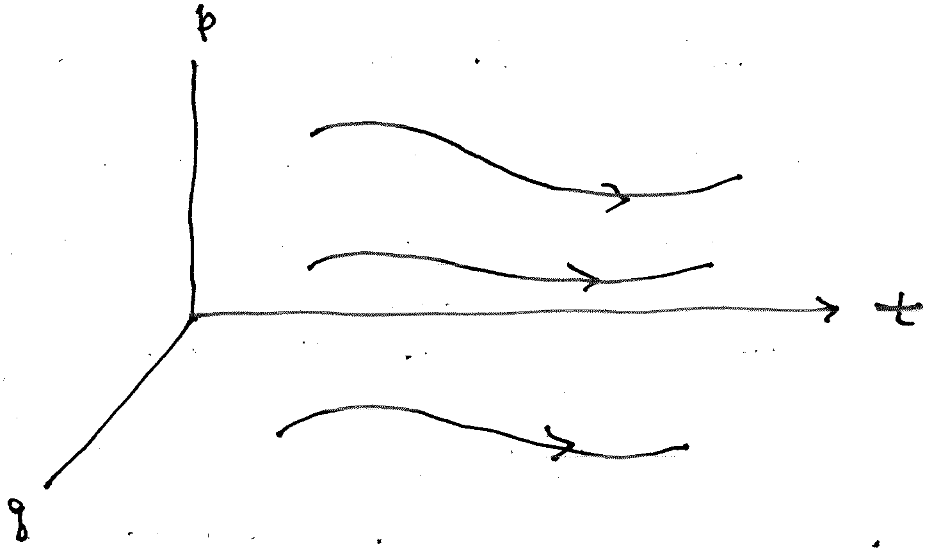
$$\begin{pmatrix} +\partial H/\partial p \\ -\partial H/\partial q \\ 1 \end{pmatrix} \tag{41}$$

which implies Hamilton's equations.

In general, $\ker\omega$ is a vector subspace of each tangent space on whatever manifold ω is defined on. If the rank of ω $(=2n)$ is constant, then $\ker\omega$ is a k-dimensional <u>distribution</u>, where $2n+k=$ dim of the manifold. For example, (40) specifies a 2-form $\omega = d(p\,dq - H\,dt)$ which has rank 2 everywhere on $T^*M\times\mathbb{R}$ (a 3d space), that is, $\dim\ker\omega = 1$ everywhere. This a 1-dimensional distribution, and it looks like this:



A 1-dimensional distribution on any space is integrable, and the integral manifolds are similar to integral curves of a vector field. In fact, if we choose a smooth vector field X lying in the distribution, then the integral curves of X <u>are</u> the integral manifolds of the distribution. In this example on $T^*M\times\mathbb{R}$, the are the classical orbits in the time-extended phase space:



The distribution itself does not give a privileged parameterization of the curves; the integral manifolds of a 1-d distribution are unparameterized curves. But t is one of the coordinates in $T^*M \times \mathbb{R}$, and if we ~~p~~ use t as a parameter, we get the orbit $q(t), p(t)$ in the usual sense.

Now we give some generalities about closed 2-forms and ~~there~~ their kernels. Let $\omega \in \Omega^2(M)$, $d\omega = 0$, any M. ~~Then~~ Let $N = \dim M$, and let $\operatorname{rank} \omega|_x = 2n$, $x \in M$. Then $2n \leq N$, and $\dim \ker \omega|_x = N - 2n \equiv k$. Then k is in general a function of position within M.

Consider a region of M of full dimensionality in which k is constant. Then $\ker \omega|_x$ is a k-dimensional distribution. Is this distribution integrable, in the sense of Frobenius, i.e., ~~are these~~ does M foliate locally into integrable manifolds?

Let $X, Y \in \mathfrak{X}(M)$, with $X, Y \in \Delta$ (that is, at each x, $X|_x, Y|_x$ lie in the k-dimensional subspace $\ker\omega|_x \subset T_xM$). Let $Z \in \mathfrak{X}(M)$ be an arbitrary vector field. Then since $d\omega = 0$, we have

$$0 = d\omega(X,Y,Z) = X\omega(Y,Z) - Y\omega(X,Z) + Z\omega(X,Y)$$
$$-\omega([X,Y],Z) + \omega([X,Z],Y) - \omega([Y,Z],X). \qquad (42)$$

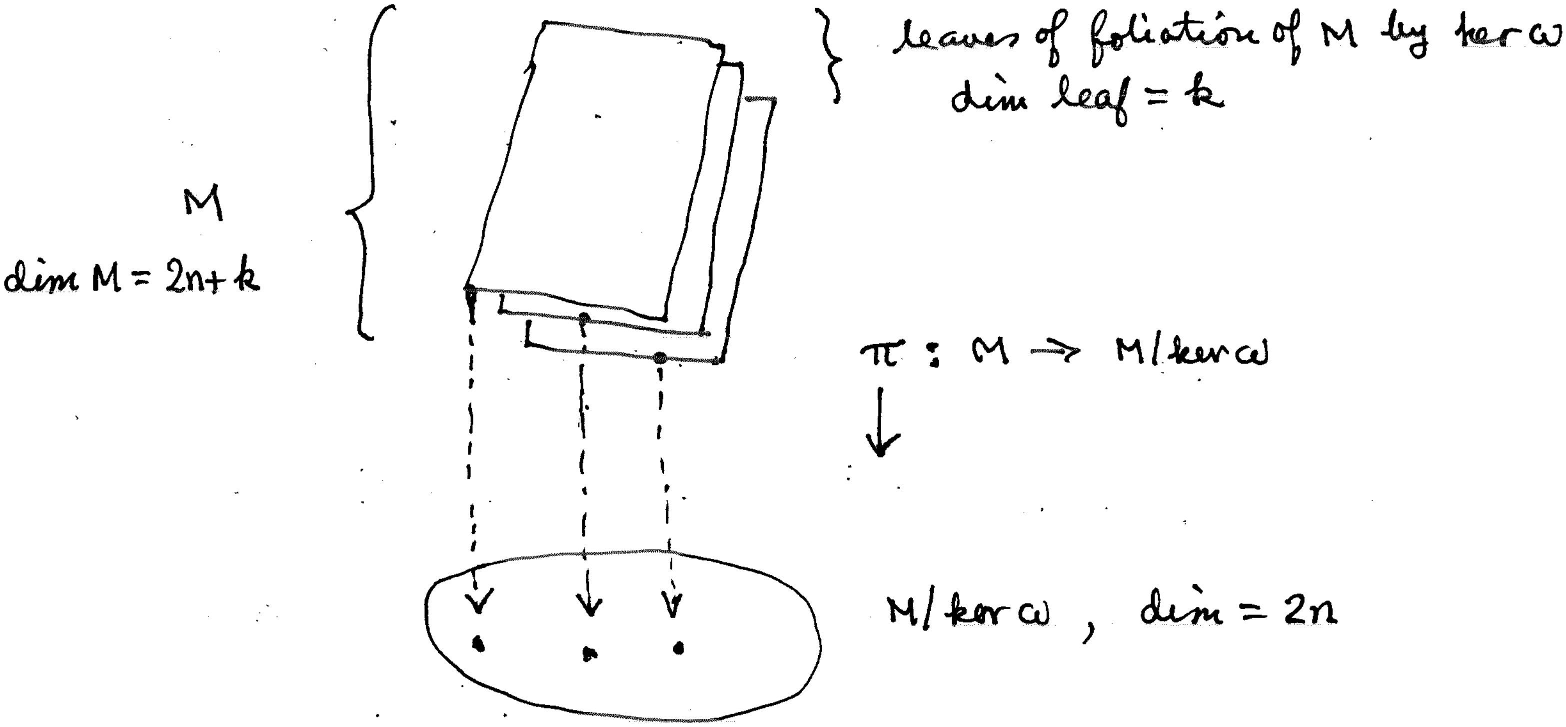
But since $X, Y \in \ker\omega$, all terms on the RHS vanish except $\omega([X,Y],Z)$. So we obtain

$$\omega([X,Y],Z) = 0 \qquad \forall Z \qquad (43).$$

$$\Rightarrow \quad [X,Y] \in \ker\omega. \qquad (44)$$

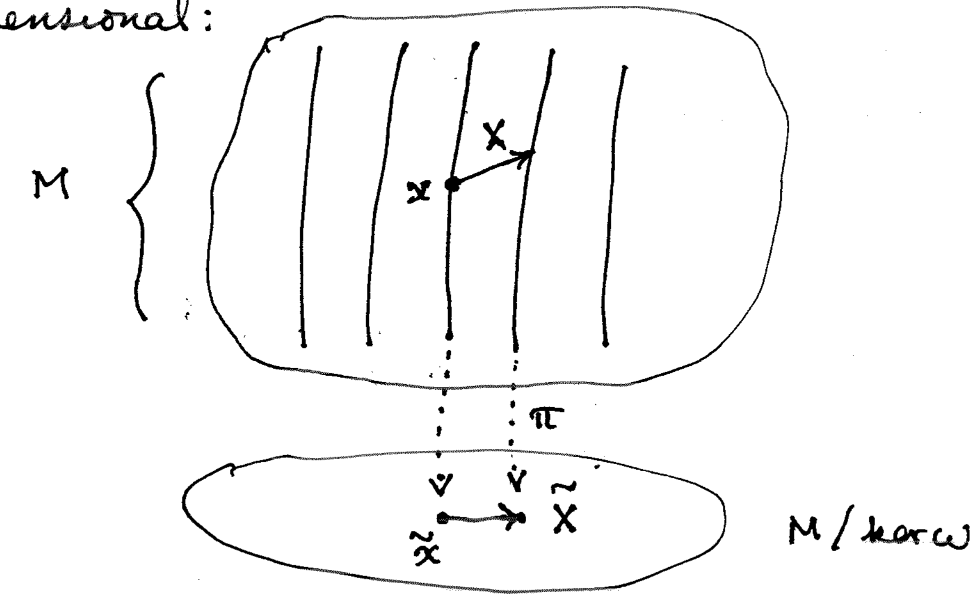
So the conditions of the Frobenius theorem are met, and $\Delta = \ker\omega$ is integrable.

So a manifold M with $\omega \in \Omega^2(M)$, $d\omega = 0$, and with $\mathrm{rank}\,\omega_{\mu\nu}$ indep. of $x \in M$, is locally foliated into integral manifolds of $\ker\omega$. In nice cases the foliation is global. If so, we can divide M by the foliation, that is, set $x \sim x'$ if x, x' lie on the same leaf of the foliation. In this way we obtain a quotient space $M/\ker\omega$, which has dimension $2n$ (where $\mathrm{rank}\,\omega_{\mu\nu} = 2n$, $\dim\ker\omega_{\mu\nu} = k$, and $2n+k = \dim M$).

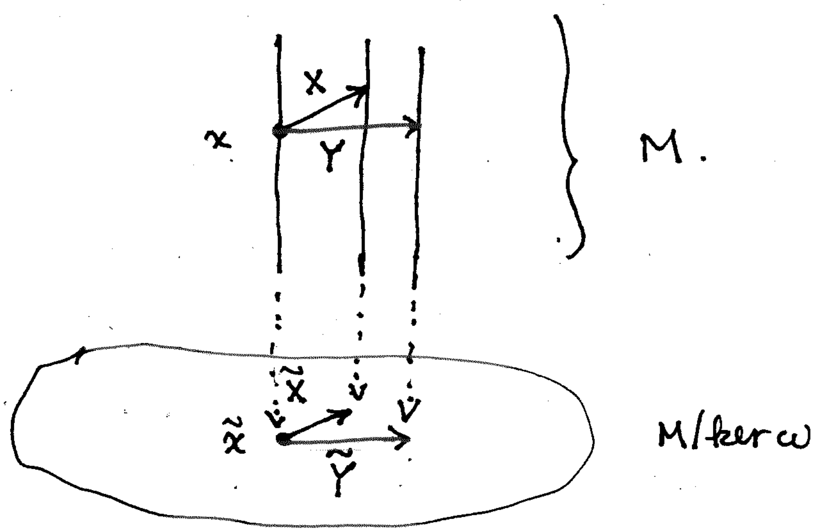


Each point of $M/\ker\omega$ represents a k-dimensional leaf of the foliation.

Then it becomes possible to effectively "project" the 2-form ω on M onto a new 2-form $\tilde{\omega}$ on $M/\ker\omega$, which turns out to be symplectic. ~~Let $x \in M$ be a point, let $\tilde{x}$ be its projection $\tilde{x} = \pi x$, and let $\tilde{X}$ be a vector tangent to $M/\ker\omega$ at $\tilde{x}$.~~ To define $\tilde{\omega}$, let $\tilde{x} \in M/\ker\omega$, and let x be a chosen point on the leaf over $\tilde{x}$, so that $\pi x = \tilde{x}$. Then let $\tilde{X}$ be a vector tangent to $M/\ker\omega$ at $\tilde{x}$. We can intuitively think of $\tilde{X}$ as connecting two neighboring points of $M/\ker\omega$, which correspond to two neighboring leaves in M. Then let X be a vector tangent to M at x that connects the two neighboring leaves, so that $\pi_* X = \tilde{X}$. Here is a diagram in which the leaves are sketched as if they were one-dimensional:



Now we do the same thing for a second vector $\tilde{Y}$:

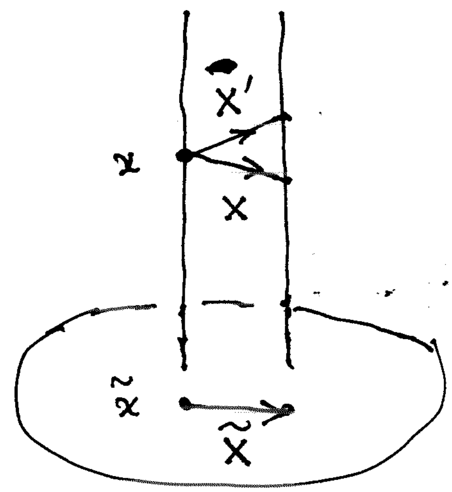


Now we wish to define $\tilde{\omega} \in \Omega^2(M/\ker\omega)$ by

$$\tilde{\omega}\big|_{\tilde{x}}(\tilde{X},\tilde{Y}) = \omega\big|_x (X,Y) \quad (?) \qquad (45)$$

but we put a ? beside it because we have to check that the answer is independent of the choice of x on the leaf over $\tilde{x}$, and ~~also~~ independent of the choice of X and Y in T_xM

(X and Y must connect x to specified nearby leaves, but the position of the tip on the leaf can be moved around so X and Y are not unique). Here are 2 vectors $X, X' \in T_x M$ that project onto the same $\tilde{X} \in T_{\tilde{x}}(M/\ker\omega)$:



$$\pi_* X = \pi_* X' = \tilde{X}$$

Vectors X, X' differ by a vector $Z \in \ker\omega$,

$$X' = X + Z, \quad Z \in \ker\omega. \tag{46}$$

But
$$\omega|_x (X+Z, Y) = \omega|_x (X, Y), \tag{47}$$

so the answer doesn't depend on which vector connects the two nearby leaves. This is because $\ker\omega|_x = \ker\pi_*|_x$.

Next we need to show that the answer doesn't depend on the choice of x on the leaf $\pi^{-1}(\tilde{x})$. Let us think of the small parallelogram spanned by X and Y at $x \in M$ as a small 2-chain c_0 on M. Then

$$\omega|_x (X, Y) = \int_{c_0} \omega \tag{48}$$

Really the parallelogram is the image of the chain. Now let Z

be a vector field on M that lies in $\ker\omega$, and let Φ_t be the advance map of the Z-flow. Also let $c_t = \Phi_t \circ c_0$. So c_t is a small parallelogram based at $\Phi_t x$ which is on the same leaf as x since $Z \in \ker\omega$. And the transported vectors $\Phi_{t*}X$, $\Phi_{t*}Y$ connect the same nearby leaves as do X and Y. Now use our formula,

$$\frac{d}{dt}\int_{c_t} \omega = \int_{c_t} i_Z d\omega + \int_{c_t} d i_Z \omega . \qquad (49)$$

The first integral vanishes, because $d\omega = 0$, and the second because $i_Z\omega = 0$. So, the integral $\int_{c_t}\omega$ is independent of t,

and thus $\omega|_x(X,Y)$ is independent of x, as long as any x on the leaf $\pi^{-1}(\tilde{x})$ can be reached from any other point by the flow of a vector field. This will be the case if the leaves are connected.

~~Ultimately, it is easy to show that~~

Thus the formula (45) does define a 2-form $\tilde{\omega} \in \Omega^2(M/\ker\omega)$. With some extra effort we can show that it is closed and nondegenerate, that is, $\tilde{\omega}$ is a symplectic form on $M/\ker\omega$.

In summary, this construction requires that $\operatorname{rank}\omega_{\mu\nu}$ be constant on M; that foliation of M by $\ker\omega$ be global; and that the leaves be connected. Under these circumstances, $M/\ker\omega$

is a symplectic manifold.

For example, on the space $T^*M \times \mathbb{R}$, with coordinates (q, p, t), we had the 1-form

$$\theta = p\,dq - H\,dt \tag{50}$$

and the presymplectic 2-form,

$$\omega = dp \wedge dq - dH \wedge dt, \tag{51}$$

where the orbits are curves in this space whose tangent vectors X satisfy $i_X \omega = 0$. Here $\operatorname{rank} \omega_{\mu\nu} = 2$ and $\dim \ker \omega = 1$, so the orbits are the 1-dimensional integral manifolds. Then the quotient space $M/\ker\omega$ is 2d and is symplectic. It is the space in which each point represents an entire orbit of the system, from $t = -\infty$ to $t = +\infty$.

Each orbit punctures each q-p plane at fixed t at a single point, so that point can be taken as a representative element of the equivalence class. In this way, the space $M/\ker\omega$ becomes identified with a q-p plane at fixed t. But the choice of t is just a convention for the representative elements. Actually the points of $M/\ker\omega$ represent the "state" of the classical system in the same way that the "state" of a quantum system is represented in the Heisenberg picture.