$$\langle \alpha, \Delta\alpha \rangle = \langle \alpha, d^\dagger d\alpha \rangle + \langle \alpha, dd^\dagger \alpha \rangle$$
$$= \langle d\alpha, d\alpha \rangle + \langle d^\dagger \alpha, d^\dagger \alpha \rangle \geq 0$$

since both terms are ~~pos~~ ≥ 0. In fact, because $\langle\, , \rangle$ is pos. def., we have more:

$$\langle \alpha, \Delta\alpha \rangle = 0 \quad \text{iff} \quad d\alpha = 0 \text{ and } d^\dagger\alpha = 0.$$

In fact, there is more than this. If $d\alpha = 0$ and $d^\dagger\alpha = 0$, then $d^\dagger d\alpha = 0$ and $dd^\dagger\alpha = 0$, so $\Delta\alpha = 0$. But $\Delta\alpha = 0 \Rightarrow \langle \alpha, \Delta\alpha \rangle = 0 \Rightarrow d\alpha = 0$ and $d^\dagger\alpha = 0$.

So altogether we have

$$\boxed{\langle \alpha, \Delta\alpha \rangle = 0 \iff \begin{pmatrix} d\alpha = 0 \text{ and} \\ d^\dagger\alpha = 0 \end{pmatrix} \iff \Delta\alpha = 0}$$

Now we make some definitions.

A form $\omega \in \Omega^r(M)$ is

<u>closed</u>	if	$d\omega = 0$	
<u>coclosed</u>	if	$d^\dagger\omega = 0$	
<u>exact</u>	if	$\omega = d\psi$,	some $\psi \in \Omega^{r-1}(M)$
<u>coexact</u>	if	$\omega = d^\dagger\psi$,	some $\psi \in \Omega^{r+1}(M)$
<u>harmonic</u>	if	$\Delta\omega = 0$	

Note that by thm. above ω is harmonic if and only if it is both closed and coclosed. Actually there there is a interesting set of relationships among the spaces of the different kinds of forms. Define:

C = { closed r-forms } = $Z^r(M)$

CC = { coclosed r-forms }

E = { exact r-forms } = $B^r(M) \subseteq Z^r(M)$

CE = { coexact r-forms }

H = { harmonic r-forms } = $\mathrm{Harm}^r(M)$.

Then it turns out that $\Omega^r(M)$ can be decomposed into 3 orthogonal subspaces:

$$\boxed{\Omega^r(M) = E \oplus CE \oplus H}$$

Proof: First show that spaces are orthogonal.

(a) prove $\langle E, CE\rangle = 0$. Let $\alpha = d\psi$, $\beta = d^\dagger\phi$ $(\alpha, \beta \in \Omega^r(M))$.

Then $\langle \alpha, \beta\rangle = \langle d\psi, d^\dagger\phi\rangle = \langle \psi, d^\dagger d^\dagger \phi\rangle = 0$.

(b) prove $\langle E, H\rangle = 0$. Let $\alpha = d\psi$, $\Delta\beta = 0$. Then

$$\langle \alpha, \beta\rangle = \langle d\psi, \beta\rangle = \langle \psi, d^\dagger\beta\rangle = 0$$

since $\Delta\beta = 0 \Rightarrow d^\dagger\beta = 0$.

(c) prove $\langle CE, H\rangle = 0$. Let $\alpha = d^\dagger\psi$, $\Delta\beta = 0$. Then

$$\langle \alpha, \beta\rangle = \langle d^\dagger\psi, \beta\rangle = \langle \psi, d\beta\rangle = 0$$

since $\Delta\beta = 0 \Rightarrow d\beta = 0$.

Next show that $E \oplus CE \oplus H$ is the entire space $\Omega^r(M)$, by showing that if $\omega \in \Omega^r(M)$ is orthogonal to E, CE, and H, then $\omega = 0$. This is a completeness proof. ~~Let $\alpha \in E$, $\beta \in CE$, $\gamma \in H$,~~ Suppose

(a) $\langle \omega, \alpha \rangle = 0 \quad \forall\, \alpha \in E$, i.e., $\forall\, \alpha$ such that $\alpha = d\psi$

(b) and $\langle \omega, \beta \rangle = 0 \quad \forall\, \beta \in CE$, i.e., $\forall\, \beta$ such that $\beta = d^\dagger \phi$

(c) and $\langle \omega, \gamma \rangle = 0 \quad \forall\, \gamma \in H$, i.e., $\forall\, \gamma$ such that $\Delta \gamma = 0$.

(a) $\Rightarrow \quad \langle \omega, d\psi \rangle = 0 \quad \forall\, \psi \in \Omega^{r-1}(M)$

$\Rightarrow \langle d^\dagger \omega, \psi \rangle = 0 \Rightarrow d^\dagger \omega = 0.$

(b) $\Rightarrow \langle \omega, d^\dagger \phi \rangle = 0 \quad \forall\, \phi \in \Omega^{r+1}(M)$

$\Rightarrow \langle d\omega, \phi \rangle = 0 \Rightarrow d\omega = 0$

(c) (a) and (b) $\Rightarrow \Delta\omega = 0$, so (c) $\Rightarrow \langle \omega, \omega \rangle = 0$ (by $\gamma = \omega$)

$\Rightarrow \omega = 0.$

QED.

See drawing next page.

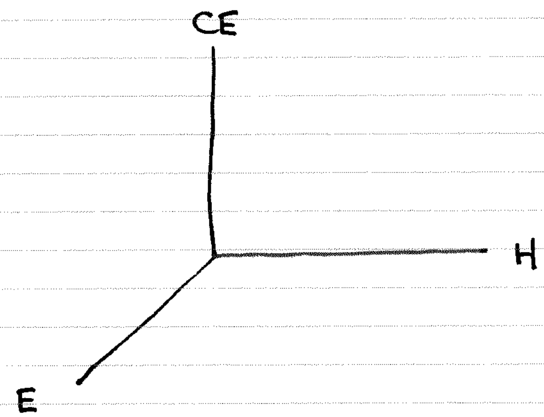
Here are more relations. We know that $E \subseteq C$ $\quad (B^r(M) \subseteq Z^r(M))$. It turns out that $C = E \oplus H$. Similarly, $CC = CE \oplus H$

Proof that $C = E \oplus H$. Let $\langle \alpha, CE \rangle = 0$, i.e., $\langle \alpha, d^\dagger \beta \rangle = 0$, $\forall\, \beta \in \Omega^{r+1}(M)$. This implies $\langle d\alpha, \beta \rangle = 0$, $\forall\, \beta$, $\Rightarrow d\alpha = 0$. Conversely, $d\alpha = 0 \Rightarrow \langle d\alpha, \beta \rangle = 0$, $\forall\, \beta \Rightarrow \langle \alpha, d^\dagger \beta \rangle = 0$, $\forall \beta \Rightarrow \langle \alpha, CE \rangle = 0$. So α is orthogonal to all co-exact forms iff α is closed. This means,

$$\boxed{\begin{aligned} C &= E \oplus H \\ CC &= CE \oplus H \end{aligned}}$$

Sim.

The Hilbert space $\Omega^r(M)$:



$C =$ "E–H plane"

$CC =$ "CE–H plane".

Hence $H = C \cap CC$ as noted above.

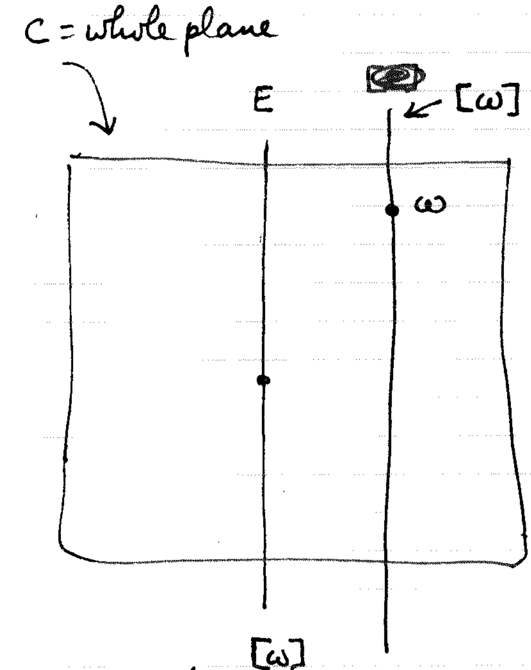
From this follows a theorem. An arbitrary form $\omega \in \Omega^r(M)$ has a unique decomposition,

$$\omega = \alpha + \beta + \gamma$$

where $\alpha = d\psi$, $\beta = d^\dagger \phi$, $\Delta \gamma = 0$, i.e.,

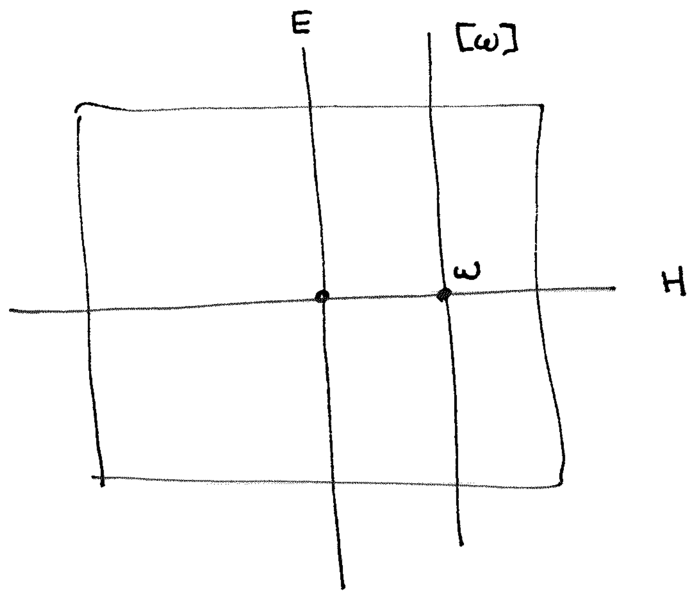
$$\omega = d\psi + d^\dagger \phi + \gamma.$$

Finally, there are some connections with cohomology theory. Recall, an element of $H^r(M)$ is an equivalence class $[\omega] = [\omega + d\psi]$ of closed forms, $d\omega = 0$. Look at the geometry of the space $Z^r(M) = C$ before we put in a metric. It's just a vector space, sketch here as 2D, with a subspace of exact forms $E = B^r(M)$, sketch here as 1D subspace:



Then an equivalence class is seen geometrically as a plane (line) parallel to E, containing representative element ω, see picture.

Now when we add a metric, we can talk about the orthogonal space in C, which is $H = \mathrm{Harm}^r(M)$:



and there appears a privileged choice for a representative element of ~~an element of $H^r(M)$~~, a cohomology class, namely, a harmonic form:

> Every cohomology class $\in H^r(M)$ contains a unique Harmonic form ω. This is the form in the cohomology class that minimizes $\langle \omega, \omega \rangle$.

An immediate corollary is that $\mathrm{Harm}^r(M)$ is isomorphic to $H^r(M)$,

$$\boxed{\mathrm{Harm}^r(M) \cong H^r(M)}$$

Every harmonic form corresponds to a unique cohomology class, and vice-versa.

The space of harmonic forms is otherwise the space of eigen-forms of Δ with eigenvalue 0. Thus, the Betti number satisfies

$$b_r = \dim H^r(M) = \text{degeneracy of eigenvalue 0 of } \Delta \text{ acting on } \Omega^r(M).$$

Examples. For a zero form, $\Delta f = 0 \Rightarrow df = 0 \Rightarrow f = \text{const}$, and conversely. Thus (assuming M is connected), $\mathrm{Harm}^0(M)$ is spanned by $f=1$, it is one-dimensional, and $b_0(M) = 1$, which we knew already.

For case $r=1$, take some sample spaces.

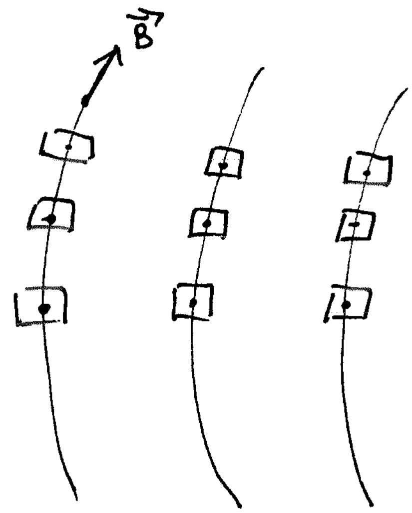
S^1 = Circle, $\Delta\omega = 0 \Rightarrow \omega = d\theta$. $\quad b_1 = 1$

T^2 Torus, $\Delta\omega = 0 \Rightarrow \omega = d\theta_1, d\theta_2$ $\quad b_1 = 2$

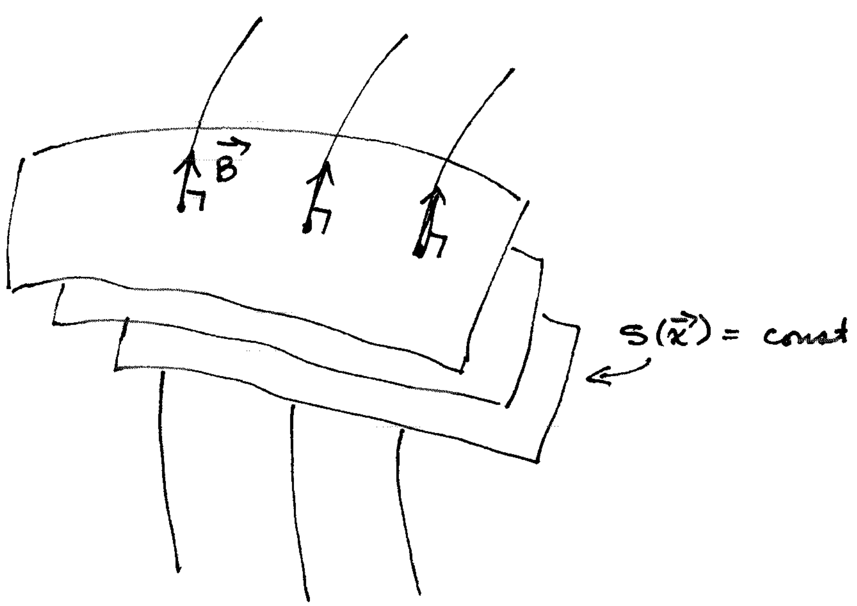
S^2 = sphere $\Delta\omega = 0 \Rightarrow \omega = 0$. $\quad b_1 = 0$.

Now we take up the Frobenius theorem, a basic result in differential geometry. On any manifold M, we define an <u>r-distribution</u> Δ as a smooth assignment of r-dimensional subspaces $(0 \leq r \leq \dim M)$ in each tangent space $T_x M$.

For example, consider $M \subseteq \mathbb{R}^3$, a region in which a nonzero magnetic field $\vec{B}$ is defined. The field has field lines (the integral curves of $\vec{B}$). At each point of this region we consider the planes (or small pieces of planes) $\perp$ to $\vec{B}$.



The planes really live in the tangent spaces. A natural question is whether these little pieces of planes can be glued together smoothly to form surfaces (actually, a family of surfaces). If so, $\vec{B}$ is everywhere $\perp$ to the surfaces.



By assigning values to these surfaces (const on each surface), we can regard the surfaces as level sets of a scalar $s(\vec{x})$. Then since $\vec{B} \perp$ surfaces, we have

$$\vec{B} = \mu \nabla s$$

where μ is some scalar (it can depend on $\vec{x}$). But this implies

$$\nabla \times \vec{B} = \nabla\mu \times \nabla s,$$

$$\vec{B} \cdot (\nabla \times \vec{B}) = 0.$$

So such surfaces exist only if $\vec{B} \cdot \nabla \times \vec{B} = 0$. In fact, it is iff, $\vec{B} \cdot \nabla \times \vec{B} = 0$ is the integrability condition for the existence of scalars μ and s in $\vec{B} = \mu \nabla s$, given $\vec{B}$. We see that in general, 2D surfaces orthogonal to a vector field in 3D do not exist.

We say that an r-distribution Δ on M is <u>integrable</u> if there exists (locally) a foliation of M into r-dimensional submanifolds such that at each x (in some local region of M) Δ_x is tangent to the manifold passing through x.

We say that a vector field $X \in \mathfrak{X}(M)$ lies in a distribution Δ if $X|_x \in \Delta|_x$ at each x. If Δ is integrable, then ~~the~~ any integral curves of $X \in \Delta$ must lie in one of the submanifolds to which Δ is tangent (a fairly obvious geometrical fact). Thus, by following integral curves of vector fields lying in an integrable distribution Δ, we can explore the corresponding submanifolds. ~~These~~ Such vector fields need not commute, but if the distribution is integrable, then $[X, Y]$ should lie in Δ. (Another fairly intuitive fact.) These ideas make plausible the following theorem:

<u>Thm</u>. A distribution Δ is integrable iff $X, Y \subset \Delta$ (X, Y vector fields on M) implies

$$[X, Y] \subset \Delta.$$

This is the <u>Frobenius</u> <u>theorem</u>. It has several versions. If we specify an r-distribution by r linearly indep. vector fields $X_1, \ldots, X_r$, then Δ is integrable iff

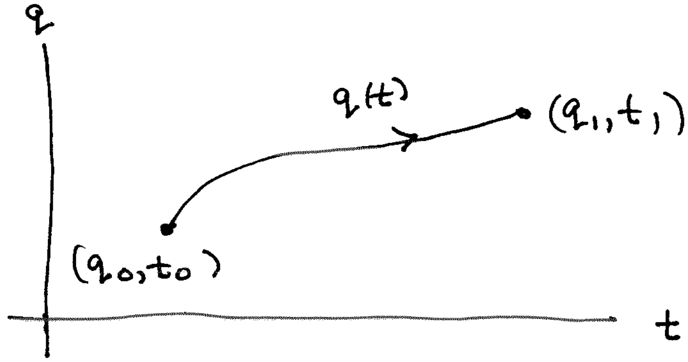
$$[X_i, X_j] = C_{ij}^k X_k,$$

where the C_{ij}^k are allowed to be functions of position. (Thus, the $\{X_i\}$ do not usually form a Lie algebra.)

To return to connections, we can say that a connection on a P.F.B. is an m-dimensional distribution on P ($m = \dim M$), invariant under the group action and transverse to the fibers.

In general, this distribution which defines a connection is not integrable. One can show that it is integrable iff the curvature tensor (on M) vanishes.

Some notes on the geometry of variational principles. First consider Lagrangian mechanics. To set up the variational principle, we start with two points in the q-t plane,



and paths $q(t)$ that connect them. We define a path space,

$$\mathcal{P} = \{\, q(t) \mid q(t_0) = q_0,\ q(t_1) = q_1 \}$$

smooth

The the action is a map,

$$A : \mathcal{P} \to \mathbb{R} : q(t) \longmapsto \int_{t_0}^{t_1} L(q(t), \dot{q}(t), t)\, dt \tag{1}$$

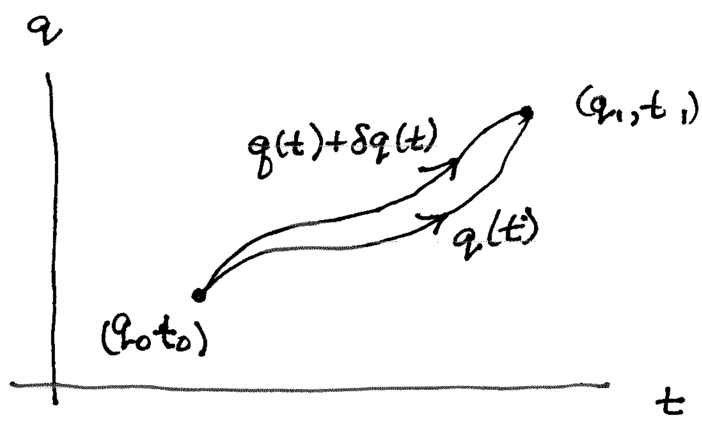
where $L(q, \dot{q}, t)$ is the Lagrangian. The paths $q(t)$ that we plug into (1) need not be physical paths, which are defined as solutions of the classical equations of motion; in general we will call them "test functions".

But Hamilton's principle states that $q(t)$ is physical iff

$$\frac{\delta A}{\delta q(t)} = 0 \tag{2}$$

or $\delta A = 0$ to first order in $\delta q(t)$. Since the domain of A is $\mathcal{P}$, both a path $q(t)$ and a nearby path $q(t) + \delta q(t)$

must belong to $\mathcal{P}$; this means $\delta q(t_0) = \delta q(t_1) = 0$.



Then $\delta A = A[q(t) + \delta q(t)] - A[q(t)]$,

or $$\delta A = \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot q}\delta \dot q \right) dt$$

$$= \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right)\right)\delta q \quad + \quad \frac{\partial L}{\partial \dot q}\delta q \Big|_{t_0}^{t_1}, \qquad (3)$$

where the last term vanishes because of the endpoint conditions on δq. Then demanding that $\delta A = 0$ for all δq gives the Euler-Lagrange equations,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right) = \frac{\partial L}{\partial q}. \qquad (4)$$

This has been done for one degree of freedom (a single q), but the results are easily generalized. More generally the q-axis in the diagrams becomes configuration space, call it M, with coordinates q^i, and the "q-t plane" becomes $M \times \mathbb{R}$, where $\mathbb{R}$ refers to time. We can call $M \times \mathbb{R}$ "the time-extended configuration space."

In most mechanical problems, the Euler-Lagrange equations (4) can be solved for the accelerations, giving equations of the form

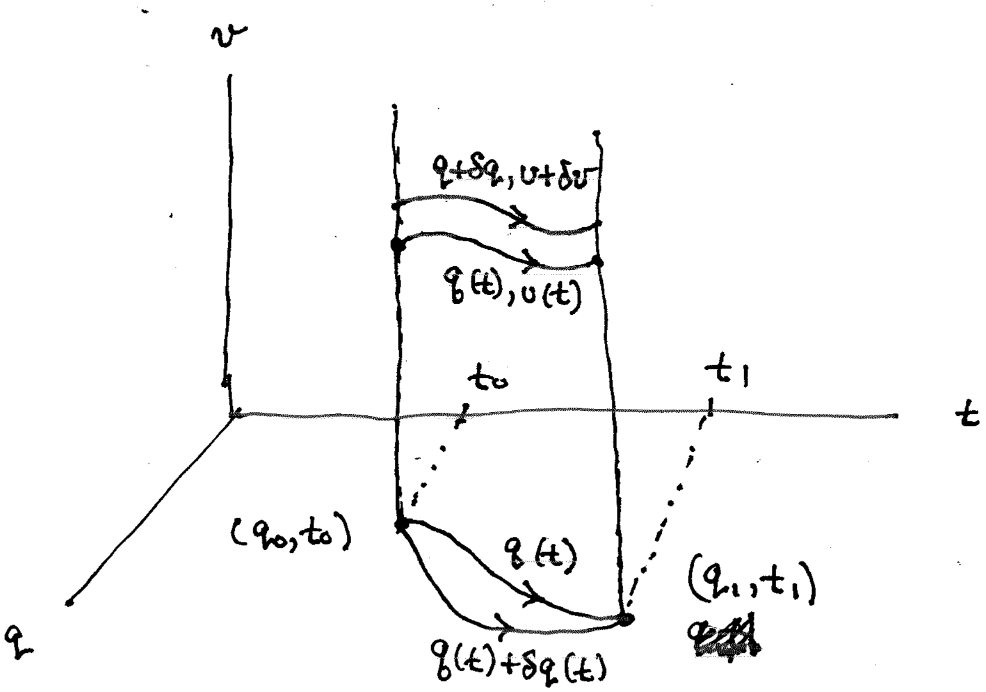
$$\ddot{q} = F(q, \dot{q}, t) \tag{5}$$

for some function F. This is not a vector field on M, which would mean a system of 1st order differential equations.

But we can make the equations first order if we write $\dot{q} = v$, and replace (5) by the system

$$\left.\begin{aligned} \dot{q} &= v \\ \dot{v} &= F(q, v, t) \end{aligned}\right\} \tag{6}$$

The solutions $q(t), v(t)$ of these equations can be visualized in qvt-space (geometrically, $TM \times \mathbb{R}$), as shown:



The q-t plane in this picture is the same as before, but now there is a v-axis.

Given a path $q(t)$ in the qt-plane, we can differentiate to get $v(t) = \dot{q}(t)$, and plot the curve $q(t), v(t)$ in qvt-space. The latter is called the <u>lift</u> of the curve $q(t)$. In the picture are shown curves $q(t)$ and $q(t)+\delta q(t)$ in the q-t plane, and their lifts, $q(t), v(t)$ and $q+\delta q, v+\delta v$, where $\delta v = \delta\dot{q}$. Although $q(t)$ and $q(t)+\delta q(t)$ coincide at the endpoints, the lifted curves do not, since δv is not required to vanish at the endpoints.

If we draw an arbitrary curve $q(t), v(t)$ in qvt-space, it is not necessarily the lift of a curve $q(t)$ in qt-space. It will be so if it satisfies the <u>lift-condition</u> $\dot{q}(t) = v(t)$. Of course one of the equations of motion (6) is precisely the lift condition.

The lift condition can also be stated in terms of differential forms. Consider the form,

$$dq - v\,dt \in \Omega^1(qvt\text{-space}). \tag{7}$$

This is defined everywhere on qvt-space. Then a curve γ in qvt-space is a lift iff $(dq - v\,dt)(X) = 0$, for any X tangent to γ. In other words, regarding γ as a 1d submanifold of qvt-space, we have $(dq - v\,dt)\big|_\gamma = 0$.

Can the lifted equations of motion (6) be expressed in variational form? Yes, if we enforce the lift condition by means of a Lagrange multiplier. Along side the original action,

$$A[q(t)] = \int_{t_0}^{t_1} L(q,\dot{q},t)\,dt \tag{8}$$

let us put down a lifted action

$$\bar{A}[q(t), v(t), p(t)] = \int_{t_0}^{t_1} L(q,v,t)\,dt + p(\dot{q}-v)\,dt \tag{9}$$

where the bar on $\bar{A}$ means "lifted" and where $p(t)$ is the Lagrange multiplier. We can write the integrand of (9) as $\bar{L}\,dt$, where $\bar{L}$, the lifted Lagrangian, is

$$\begin{aligned} \bar{L}(q,v,p;\dot{q},\dot{v},\dot{p};t) &= L(q,v,t) + p\dot{q} - pv \\ &= p\dot{q} - H(q,v,p,t), \end{aligned} \tag{10}$$

where H is defined by

$$H(q,v,p,t) = pv - L(q,v,t). \tag{11}$$

This is the usual definition of the Hamiltonian, except that it depends on v (as an independent variable) instead of just (q,p,t) as in the usual Hamiltonian mechanics.

We can think of $\bar{L}$ in (10) as a function of $\dot{q}, \dot{v}, \dot{p}$, as indicated, except that it happens that $\dot{v}$ and $\dot{p}$ don't appear. Thus, the Euler-Lagrange equations are

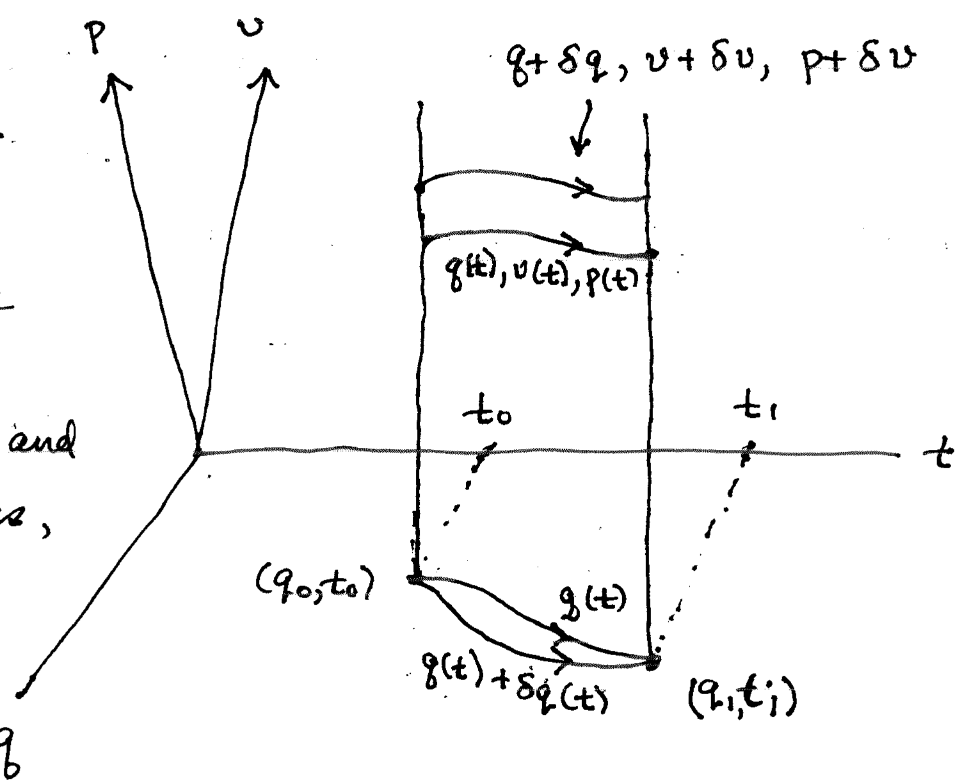
$$\frac{d}{dt}\frac{\partial \bar{L}}{\partial \dot{q}} = \frac{\partial \bar{L}}{\partial q} \qquad \text{or} \qquad \dot{p} = \frac{\partial L}{\partial q} \tag{12a}$$

$$\frac{d}{dt}\frac{\partial \bar{L}}{\partial \dot{v}} = \frac{\partial \bar{L}}{\partial v} \qquad \text{or} \qquad p = \frac{\partial L}{\partial v} \tag{12b}$$

$$\frac{d}{dt}\frac{\partial \bar{L}}{\partial \dot{p}} = \frac{\partial \bar{L}}{\partial p} \qquad \text{or} \qquad \dot{q} = v \tag{12c}$$

Of course (12c) is the lift condition; we set up $\bar{L}$ to make that come out. And (12b) shows that p is the momentum conjugate to q, according to the usual definition and as suggested by the notation. For regular Lagrangians (12b) gives p as an invertible function of v, which is the Legendre transformation. Notice that it is an algebraic equation, not a differential equation.

To represent the lifted variational principle geometically, we need a space in which both p and v are coordinates, in addition to q and t. This space is



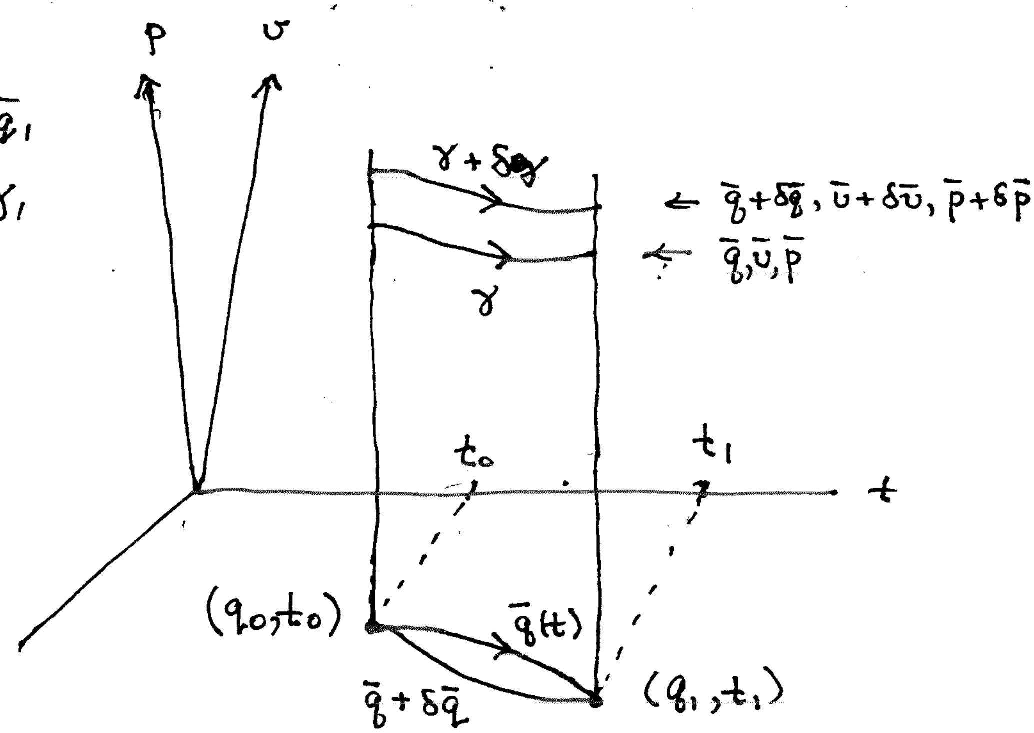
$B \times \mathbb{R}$, where $\mathbb{R}$ represents time and B is a bundle over configuration space M in which the fiber over $x \in M$ is $T_xM \times T_x^*M$. Coordinates on one fiber are (v, p), where $v = v^i \frac{\partial}{\partial x^i}\big|_x$ and $p = p_i \, dx^i\big|_x$.

At this point there arises a danger of confusing q, v, p as coordinates on $qvpt$-space ($B \times \mathbb{R}$), and the test functions $q(t), v(t), p(t)$ that are plugged into the lifted action $\bar{A}$ of Eq. (9). Therefore let us henceforth write $\bar{q}(t), \bar{v}(t), \bar{p}(t)$ for the test functions, so the lifted action becomes

$$\bar{A}[\bar{q}(t), \bar{v}(t), \bar{p}(t)] = \int_{t_0}^{t_1} L(\bar{q}, \bar{v}, t)\,dt + \bar{p}\,(\dot{\bar{q}} - \bar{v})\,dt$$

$$= \int_{t_0}^{t_1} \bar{L}(\bar{q}, \bar{v}, \bar{p}, \dot{\bar{q}}, t)\,dt \qquad (13)$$

Let us denote the curve $\bar{q}(t), \bar{v}(t), \bar{p}(t)$ in $qvpt$-space by γ, and likewise denote the curve $\bar{q} + \delta\bar{q}$, $\bar{v} + \delta\bar{v}$, $\bar{p} + \delta\bar{p}$ by $\gamma + \delta\gamma$, as shown.



On the curve γ, we have

$$\left.\begin{aligned} q &= \bar{q}(t) \\ p &= \bar{p}(t) \\ v &= \bar{v}(t) \end{aligned}\right\} . \qquad (14)$$

We also have

$$dq = \dot{\bar{q}}\, dt \qquad (15)$$

on γ, which looks obvious but which means the following: dq and dt are 1-forms defined everywhere on $qvpt$-space, and γ can be regarded as a 1-dimensional submanifold of this space. ~~At~~ a point on γ, $\dot{\bar{q}}$ is defined, and if the differential forms dq and dt are restricted to γ then (15) holds at any such point. That is, if $x \in \gamma \subset B \times \mathbb{R}$, and X is a tangent vector at x, tangent to γ, then

$$(dq - \dot{\bar{q}}\, dt)(X) = 0. \qquad (16).$$

The integrand of the lifted action (13) is a 1-form on the time-axis (a one-dimensional manifold), but in view of the relations (14) and (15) the integral can be reformulated as an integral along γ in $qvpt$-space:

$$\bar{A}[\bar{q}(t), \bar{v}(t), \bar{p}(t)] = \int_\gamma L(q, v, t)\, dt + p\, dq - pv\, dt. \qquad (17)$$

Notice that now the overbars are removed from the integrand, which is a 1-form defined everywhere in qvpt-space,

$$\theta = L(q,v,t)dt + p\,dq - pv\,dt \tag{18}$$
$$\in \Omega^1(\text{qvpt-space}) = \Omega^1(B\times\mathbb{R}).$$

The test functions $\bar{q}(t)$, $\bar{v}(t)$, $\bar{p}(t)$ now appear in γ, the path of integration. Thus we may also write the action as $\bar{A}[\gamma]$.

One can regard the transformation of the integral (13) into (16) as one of pulling back the ~~integral~~ integrand from the time axis into $B\times\mathbb{R}$, via π^*, where

$$\pi : B\times\mathbb{R} \to \mathbb{R}$$
$$: (qvpt) \mapsto t$$

is the projection map.

The 1-form θ can also be written

$$\theta = p\,dq - H(q,v,p,t)\,dt \tag{19}$$

where H is given by (11) (and note the remark below (11)). So far we have

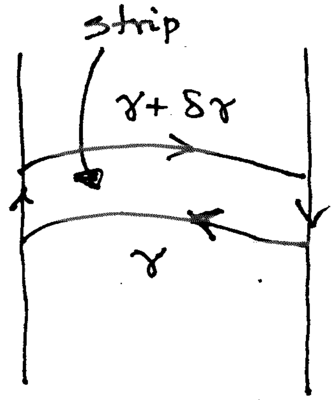
$$\bar{A}[\gamma] = \int_\gamma \theta . \tag{20}$$

Now Hamilton's principle can be put into an interesting form. Let

γ and $\gamma+\delta\gamma$ be two nearby paths, such that $\delta\bar{q}=0$ at t_0 and t_1, as in the diagrams above. Then Hamilton's principle says that γ is a physical path iff

$$\delta\bar{A} = \bar{A}[\gamma+\delta\gamma] - \bar{A}[\gamma] = 0 \qquad (21)$$

vanishes for all $\delta\gamma$ satisfying the endpoint conditions. This difference is the integral along $\gamma+\delta\gamma$ of θ in a forward direction, plus that along γ in a backward direction. But the integral of θ along the vertical segments is zero, since $dq = dt = 0$ on these segments. Thus, $\gamma(t)$ is physical if



$$\delta\bar{A} = \int_{\partial(\text{strip})} \theta = \int_{\text{strip}} \omega = 0 \qquad \text{by Stokes' theorem}$$

for all $\delta\gamma$ satisfying the endpoint conditions, where

$$\omega = d\theta = dp\wedge dq - dH\wedge dt$$

$$= dp\wedge dq - \left(\frac{\partial H}{\partial q}dq + \frac{\partial H}{\partial v}dv + \frac{\partial H}{\partial p}dp\right)\wedge dt\,, \qquad (22)$$

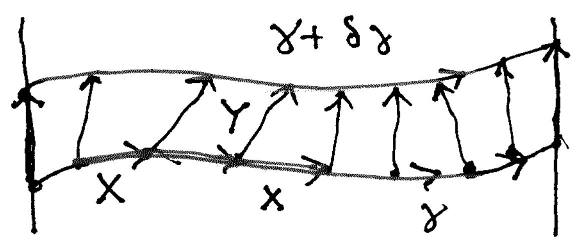
and where the "strip" is shown in the diagram. Here ω is an exact (hence closed) 2-form,

$$\omega \in \Omega^2(\text{qvpt-space}) = \Omega^2(B\times\mathbb{R}). \qquad (23)$$

This argument is intuitive rather than rigorous because we are using reasoning based on infinitesimals, but the conclusions we draw are correct. The integral of any 2-form over a 2-dimensional surface can be regarded as the limit of a Riemann sum, in which the surface is broken up into small parallelograms and the 2-form is evaluated on the small vectors spanning each parallelogram. The integral is the limit of the sum of such terms.

In the present case, the 2-dim'l area is a strip of infinitesimal width. Let Y be a vector field defined on γ, such that the base and the tip of Y connect γ and the nearby path $\gamma+\delta\gamma$.

Also, let λ be a parameter of curve γ, say, $0 \le \lambda \le 1$, so Y is a function of λ. Also let $X = d/d\lambda$ be a vector tangent to γ. Here



$$X = \frac{d\bar{q}}{d\lambda}\frac{\partial}{\partial q} + \frac{d\bar{v}}{d\lambda}\frac{\partial}{\partial v} + \frac{d\bar{p}}{d\lambda}\frac{\partial}{\partial p} + \frac{dt}{d\lambda}\frac{\partial}{\partial t}. \qquad (24)$$

(we may prefer to write $d\bar{t}/d\lambda$ instead of $dt/d\lambda$, since t is understood to be a function of λ along the curve.) Then

$$\int_{\text{strip}} \omega = \int_0^1 \omega(X, Y)\, d\lambda. \qquad (25)$$

If γ is physical, then this integral must vanish for all $\delta\gamma$, that is, all vectors $Y(\lambda)$. This can only be true if

$$\boxed{i_X \omega = 0} \tag{26}$$

This is the condition ~~for~~ that the tangent vector X to a physical path γ must satisfy. It is equivalent to the Euler–Lagrange equations, and gives them a geometrical interpretation.

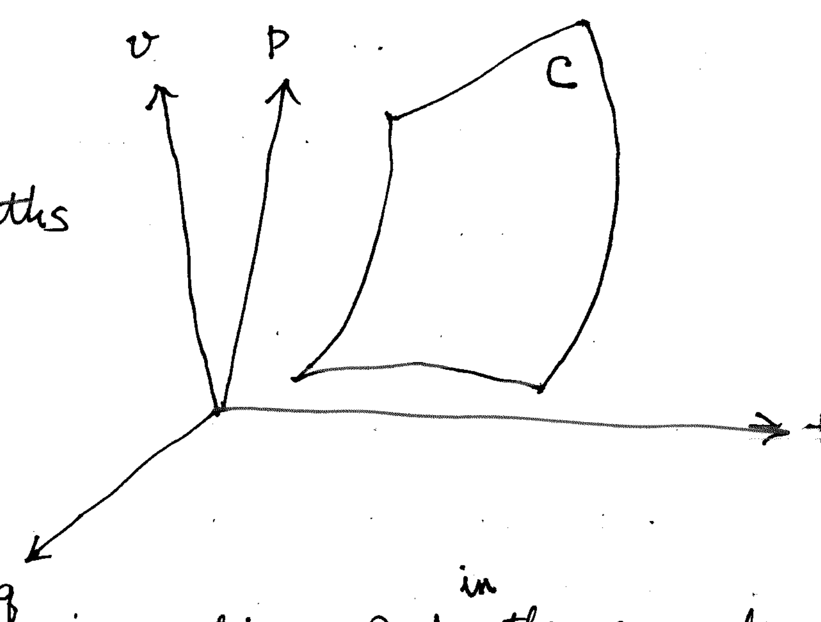
It is straightforward to put (26) into components w.r.t. the coordinates $q\,v\,p\,t$ on $B\times\mathbb{R}$, and to check that it is equivalent to the EL equations (12). This will be left as an exercise.

Instead, we make some general remarks about eqns. (26). Let $x \in B\times\mathbb{R}$, so that $\omega|_x$ can be regarded as a map from $T_x(B\times\mathbb{R})$ to $T^*_x(B\times\mathbb{R})$. Eqn. (26) says that the tangent vector to a physical path $\gamma(t)$ must lie in $\ker \omega|_x$, the kernel of this linear map. ~~Write the component matrix of~~ But $\ker \omega|_x$ is trivial (it consists only of the 0 vector) unless $\det \omega_{\mu\nu} = 0$, that is, $\operatorname{rank} \omega_{\mu\nu} < \dim(B\times\mathbb{R})$. Recall that a symplectic 2-form is one that satisfies $d\omega = 0$ and $\det \omega_{\mu\nu} \neq 0$; therefore, in this context, we do not want ω to be symplectic, because if it were then there are no solutions for X apart from $X=0$. However, we do have $d\omega = 0$. Sometimes a form with $d\omega=0$ but $\det \omega_{\mu\nu} = 0$ is called presymplectic. In the present context, we need presymplectic forms.

The component matrix $\omega_{\mu\nu}$ is antisymmetric, and the rank of an antisymmetric matrix is always even, call it $2n$. Then to have a nontrivial solution X to (26), we require $2n < N$, where $N = \dim(B \times \mathbb{R})$.

Let us drop the focus on ω for a moment and return to the Euler-Lagrange equations (12). Only (12a) and (12c) are differential equations; (12b) is purely algebraic. Whenever a purely algebraic equation emerges as one of the Euler-Lagrange equations, it specifies a constraint, that is, a submanifold of the nominal phase space on which the motion lies. In the ~~present~~ language of Dirac constraint theory, a constraint emerging from the Euler Lagrange equations is called a primary constraint.

In our example, the constraint (12b) is the submanifold C of $B \times \mathbb{R}$ upon which $p = \partial L/\partial v$. Thus C is the graph of the Legendre transformation. We can visualize it as a 3D surface in $B \times \mathbb{R} = qvpt$-space.



It is only on C that the motion takes place, that is, all physical paths γ are confined to C.

In our example, the constraint emerged cleanly as one of the Euler-Lagrange equations. But in other examples, the primary constraints ~~may~~ (the algebraic relations among the coordinates) may be tangled up with the differential equations and hard to recognize. Therefore we need a geometrical condition for the primary constraint surface.

This condition is simply ← equation of C.

$$\operatorname{rank} \omega_{\mu\nu} < \dim (B \times \mathbb{R}). \qquad (27)$$

The rank of $\omega_{\mu\nu}$ is in general a function of position, and in any case must be an even integer. In our example, where $B \times \mathbb{R}$ is $qvpt$-space and q, v, p are 1-dimensional, $\dim(B \times \mathbb{R}) = 4$. Therefore rank $\omega_{\mu\nu}$ can be either 4, 2 or 0. Working out the component matrix $\omega_{\mu\nu}$ in the coordinates $(qvpt)$ and using (22), we have

$$\omega_{\mu\nu} = \begin{array}{c c} & \begin{array}{cccc} q & v & p & t \end{array} \\ \begin{array}{c} q \\ v \\ p \\ t \end{array} & \begin{pmatrix} 0 & 0 & -1 & \partial L/\partial q \\ 0 & 0 & 0 & \frac{\partial L}{\partial v} - p \\ +1 & 0 & 0 & -v \\ -\frac{\partial L}{\partial q} & -\left(\frac{\partial L}{\partial v} - p\right) & v & 0 \end{pmatrix} \end{array} \qquad (28)$$

Actually here we have used $\theta = p\,dq - (pv - L(q,v,t))\,dt$, and

$$\omega = d\theta = dp \wedge dq + \frac{\partial L}{\partial q}\, dq \wedge dt + \left(\frac{\partial L}{\partial v} - p\right) dv \wedge dt - v\, dp \wedge dt. \qquad (29)$$

It is straightforward to show that rank $\omega_{\mu\nu} = 4$ unless $p = \partial L/\partial v$, that is, the condition (27) specifies the constraint