Now we begin Hodge $*$ theory and harmonic forms. To preview the results a bit, when we add a metric to a manifold we can do new things with differential forms and find new connections to old subjects such as cohomology groups (which do not require a metric for their definition).

If we add a metric to M, we can define a scalar product of wave functions,

$$\cancel{\langle f,g\rangle = \int_M \sqrt{|g|}\, d^m x\, fg, \quad m = \dim M}$$

$$\langle f_1, f_2\rangle = \int_M d^m x \sqrt{|g|}\; f_1 f_2$$

where $m = \dim M$, $f_1, f_2 \in \mathcal{F}(M)$. (Real valued functions here.) Thus the wave functions make a Hilbert space. We also have interesting operators that act on these wave fns, such as the generalized Laplacian ∇^2 (which requires a metric for its definition), which lead to orthonormal sets of eigenfunctions.

All this (the scalar product, Laplacians) etc. can be generalized to arbitrary r-forms. It turns out for example that the degeneracy of the 0 eigenvalue of ∇^2 is the same as the Betti number of M.

The <u>permutation</u> or <u>Levi-Civita symbol</u> is familiar:

$$\epsilon_{\mu_1 \dots \mu_m} = \begin{cases} +1 & (\mu_1 \dots \mu_m) = \text{even pmut of } (1 \dots m) \\ -1 & (\mu_1 \dots \mu_m) = \text{odd pmut of } (1 \dots m) \\ 0 & \text{otherwise} \end{cases}$$

Just because we put lower indices on it does not mean that it is a tensor. In fact, suppose a tensor has components $\epsilon_{\mu_1 \dots \mu_m}$ in one coord. system x^μ, and examine what its components are in another coord. syst. x'^μ:

$$T_{\mu_1 \cdots \mu_m} = \varepsilon_{\mu_1 \cdots \mu_m} \quad \text{in coords } x^\mu \quad \text{(suppose)}.$$

Then in coords y^μ,

$$T'_{\mu_1 \cdots \mu_m} = \frac{\partial x^{\nu_1}}{\partial y^{\mu_1}} \cdots \frac{\partial x^{\nu_m}}{\partial y^{\mu_m}} \varepsilon_{\nu_1 \cdots \nu_m} = \det\left(\frac{\partial x}{\partial y}\right) \varepsilon_{\mu_1 \cdots \mu_m}.$$

So the ε-symbol does not transform as a tensor (and we shall not call it a tensor.) Now let $g_{\mu\nu}$, $g'_{\mu\nu}$ be the metric in coords x^μ and y^μ, so that

$$g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial y^\mu} \frac{\partial x^\beta}{\partial y^\nu} g_{\alpha\beta},$$

or

$$g' = \left(\det \frac{\partial x}{\partial y}\right)^2 g \qquad \text{where} \quad \begin{array}{l} g = \det g_{\mu\nu} \\ g' = \det g'_{\mu\nu} \end{array}$$

so $\left| \det \frac{\partial x}{\partial y} \right| = \sqrt{\frac{|g'|}{|g|}}$. We put $|\ |$ around g, since it may be negative (depends on the signature, $\mathrm{sgn}(g) = -1$ in GR). Now we suppose M is orientable and we choose an orientation and work only with oriented atlases. Then $\det \frac{\partial x}{\partial y} > 0$, and we can drop the $|\ |$ around $\det \frac{\partial x}{\partial y}$. Then we see that if $T_{\mu_1 \cdots \mu_m} = \sqrt{|g|}\, \varepsilon_{\mu_1 \cdots \mu_m}$ in one coordinates, then $T'_{\mu_1 \cdots \mu_m} = \sqrt{|g'|}\, \varepsilon_{\mu_1 \cdots \mu_m}$ in another. We have a tensor, if we restrict to oriented coordinates. In fact it is an m-form, which we henceforth write as Ω:

$$\Omega = \frac{1}{m!} \sqrt{|g|}\, \varepsilon_{\mu_1 \cdots \mu_m} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_m}$$

i.e.

$$\Omega = \sqrt{|g|}\, dx^1 \wedge \cdots \wedge dx^m \qquad \text{(coord basis)}$$

or $$\Omega = \sqrt{|g|}\, \theta^1 \wedge \cdots \wedge \theta^m \qquad \text{(any basis)}.$$

Here $\{\theta^\mu\}$ is any basis (coordinate or non-coordinate). Note that if $\{\theta^\mu\}$ is an O.N. vielbein, then $\sqrt{|g|}=1$ and $\Omega = \theta^1\wedge\ldots\wedge\theta^m$.

It is of interest to compute the completely contravariant components of Ω:

$$\Omega^{\mu_1\ldots\mu_m} = g^{\mu_1\nu_1}\cdots g^{\mu_m\nu_m}\,\Omega_{\nu_1\ldots\nu_m}$$

$$= \det(g^{\mu\nu})\,\sqrt{|g|}\,\varepsilon_{\mu_1\ldots\mu_m}$$

But $\det g^{\mu\nu} = \frac{1}{g} = \mathrm{sgn}(g)/|g|$. So,

$$\boxed{\Omega^{\mu_1\ldots\mu_m} = \frac{\mathrm{sgn}(g)}{\sqrt{|g|}}\,\varepsilon_{\mu_1\ldots\mu_m}.}$$ useful later.

We don't worry that LHS has upper indices and RHS has lower, since ε is not a tensor.

Ω is called the <u>invariant volume form</u> since its integral over any region $R\subseteq M$ is the volume of that region in the metrical sense,

$$\int_R \Omega = \mathrm{vol}(R).$$

On a space with $m=\dim M$ dimensions, both r-forms and $(m-r)$-forms have the same number of indep. components,

$$\binom{m}{r} = \binom{m}{m-r}.$$

Thus r-forms and $(m-r)$-forms (at a point $x\in M$) are vector spaces of the same dimensionality, and are isomorphic as vector spaces.

In the absence of a metric or other additional structure, however, there is no <u>natural</u> isomorphism between these spaces. Now, however, we will assume we have a metric (M,g). Then there is a natural mapping (an isomorphism, actually,) between these space,

$$\text{Hodge } * : \Omega^r(M) \to \Omega^{m-r}(M).$$

It is defined by its action on the basis forms of $\Omega^r(M)$, then extended to arb. r-forms by linearity. The defn. is

$$\boxed{*(\theta^{\mu_1}\wedge\ldots\wedge\theta^{\mu_r}) = \frac{1}{(m-r)!}\,\Omega^{\mu_1\ldots\mu_r}{}_{\nu_1\ldots\nu_{m-r}}\,(\theta^{\nu_1}\wedge\ldots\wedge\theta^{\nu_{m-r}}).}$$

Indices on Ω are raised with $g^{\mu\nu}$.

As a special case, consider the 0-form $1 \in \Omega^0(M)$ (const scalar). Then $r=0$ in the above, and we have

$$*1 = \frac{1}{m!}\,\Omega_{\nu_1\ldots\nu_m}\,\theta^{\nu_1}\wedge\ldots\wedge\theta^{\nu_m} = \Omega,$$

$$\boxed{*1=\Omega}$$

The defn above makes it clear that $*$ is linear, but is it an isomorphism (i.e., is it invertible)? We answer by computing $**$, a map: $\Omega^r(M)\to\Omega^r(M)$. We apply defn above twice, get

$$**(\theta^{\mu_1}\wedge\ldots\wedge\theta^{\mu_r}) = \frac{1}{(m-r)!}\,\Omega^{\mu_1\ldots\mu_r}{}_{\nu_1\ldots\nu_{m-r}}$$

$$\times\ \frac{1}{r!}\,\Omega^{\nu_1\ldots\nu_{m-r}}{}_{\lambda_1\ldots\lambda_r}\,(\theta^{\lambda_1}\wedge\ldots\wedge\theta^{\lambda_r}).$$

Transform this. First raise + lower $\nu_1 \ldots \nu_{m-r}$ indices to make indices uniformly upper or lower. Next, on $\Omega_{\nu_1 \ldots \nu_{m-r} \lambda_1 \ldots \lambda_r}$, migrate λ indices to left of ν indices. This involves $(m-r)r$ sign changes, so

$$\Omega_{\nu_1 \ldots \nu_{m-r} \lambda_1 \ldots \lambda_r} = (-1)^{r(m-r)} \Omega_{\lambda_1 \ldots \lambda_r \nu_1 \ldots \nu_{m-r}}.$$

Thus,

$$** (\theta^{\mu_1} \wedge \ldots \wedge \theta^{\mu_r}) = \frac{1}{(m-r)!} \frac{1}{r!} (-1)^{r(m-r)}$$

$$\times \boxed{\Omega^{\mu_1 \ldots \mu_r \nu_1 \ldots \nu_{m-r}} \Omega_{\lambda_1 \ldots \lambda_r \nu_1 \ldots \nu_{m-r}}} \ \theta^{\lambda_1} \wedge \ldots \wedge \theta^{\lambda_r}$$

$$= \frac{\operatorname{sgn}(g)}{\sqrt{|g|}} \varepsilon_{\mu_1 \ldots \mu_r \nu_1 \ldots \nu_{m-r}} \times \sqrt{|g|} \ \varepsilon_{\lambda_1 \ldots \lambda_r \nu_1 \ldots \nu_{m-r}}$$

$$= \operatorname{sgn}(g) \ \operatorname{sgn}\begin{pmatrix} \mu_1 \ldots \mu_r \\ \lambda_1 \ldots \lambda_r \end{pmatrix} (m-r)!$$

where we use identities for products of two ε's and where

$$\operatorname{sgn}\begin{pmatrix} \mu_1 \ldots \mu_r \\ \lambda_1 \ldots \lambda_r \end{pmatrix} = \begin{cases} \pm 1 & \text{if } (\lambda_1 \ldots \lambda_r) \text{ is } \binom{\text{even}}{\text{odd}} \text{ perm of } \mu_1 \ldots \mu_r \\ 0 & \text{otherwise.} \end{cases}$$

Thus,

$$\operatorname{sgn}\begin{pmatrix} \mu_1 \ldots \mu_r \\ \lambda_1 \ldots \lambda_r \end{pmatrix} \theta^{\lambda_1} \wedge \ldots \wedge \theta^{\lambda_r}$$

$$= r! \ \theta^{\mu_1} \wedge \ldots \wedge \theta^{\mu_r}.$$

Putting it all together, we have

$$** \left(\theta^{\mu_1}\wedge\ldots\wedge\theta^{\mu_r}\right) = \mathrm{sgn}(g)\,(-1)^{r(m-r)} \left(\theta^{\mu_1}\wedge\ldots\wedge\theta^{\mu_r}\right).$$

or $$\boxed{** = \mathrm{sgn}(g)\,(-1)^{r(m-r)}}$$ when acting on $\omega \in \Omega^r(M)$.

equivalently, $$\boxed{*^{-1} = \mathrm{sgn}(g)\,(-1)^{r(m-r)}\, *}$$

Thus $*$ is invertible, and $*$ is an isomorphism.

Now consider the interaction of $*$ with $\wedge$. Let $\alpha, \beta \in \Omega^r(M)$. Then $*\beta$ is an $(m-r)$-form, and

$$\alpha\wedge *\beta \in \Omega^m(M).$$

Thus $\alpha\wedge *\beta$ must be proportional to the volume form Ω, i.e., it must be $f\Omega$ for some scalar f. Now we work out what f is.

Write

$$\alpha = \frac{1}{r!}\alpha_{\mu_1\ldots\mu_r}\,\theta^{\mu_1}\wedge\ldots\wedge\theta^{\mu_r}$$

$$\beta = \frac{1}{r!}\beta_{\nu_1\ldots\nu_r}\,\theta^{\nu_1}\wedge\ldots\wedge\theta^{\nu_r}$$

Hence

$$*\beta = \frac{1}{r!}\beta_{\nu_1\ldots\nu_r}\,\frac{1}{(m-r)!}\,\Omega^{\nu_1\ldots\nu_r}{}_{\lambda_1\ldots\lambda_{m-r}}\,\theta^{\lambda_1}\wedge\ldots\wedge\theta^{\lambda_{m-r}},$$

and

$$\alpha\wedge *\beta = \frac{1}{(r!)^2(m-r)!}\,\alpha_{\mu_1\ldots\mu_r}\,\beta_{\nu_1\ldots\nu_r}\,\Omega^{\nu_1\ldots\nu_r}{}_{\lambda_1\ldots\lambda_{m-r}}$$

$$\times\ \theta^{\mu_1}\wedge\ldots\wedge\theta^{\mu_r}\wedge\theta^{\lambda_1}\wedge\ldots\wedge\theta^{\lambda_{m-r}}.$$

Transform this, First raise + lower $\nu_1 \dots \nu_r$ indices, use
$\theta^{\mu_1} \wedge \dots \wedge \theta^{\mu_r} \wedge \theta^{\lambda_1} \wedge \dots \wedge \theta^{\lambda_{m-r}} = \varepsilon_{\mu_1 \dots \mu_r \lambda_1 \dots \lambda_{m-r}} \; \theta^1 \wedge \dots \wedge \theta^m$. Get,

$$\alpha \wedge * \beta = \frac{1}{(r!)^2 (m-r)!} \alpha_{\mu_1 \dots \mu_r} \beta^{\nu_1 \dots \nu_r} \sqrt{|g|} \; \varepsilon_{\nu_1 \dots \nu_r \lambda_1 \dots \lambda_{m-r}}$$

$$\times \; \varepsilon_{\mu_1 \dots \mu_r \lambda_1 \dots \lambda_{m-r}} \; \theta^1 \wedge \dots \wedge \theta^m$$

$$= \frac{1}{(r!)^2 (m-r)!} \alpha_{\mu_1 \dots \mu_r} \beta^{\nu_1 \dots \nu_r} (m-r)! \; \operatorname{sgn} \begin{pmatrix} \mu_1 \dots \mu_r \\ \nu_1 \dots \nu_r \end{pmatrix} \Omega$$

$$\boxed{\alpha \wedge * \beta = \left(\frac{1}{r!} \alpha_{\mu_1 \dots \mu_r} \beta^{\mu_1 \dots \mu_r} \right) \Omega}$$

The scalar multiplying Ω is the complete contraction of the components of α with those of β.

Several things to note about this. First, the answer is symmetric in α, β, so

$$\boxed{\alpha \wedge * \beta = \beta \wedge * \alpha}.$$

Next, if g is pos. def., then $\alpha_{\mu_1 \dots \mu_r} \alpha^{\mu_1 \dots \mu_r} \geq 0$, i.e. you get a pos. def. scalar product of r-forms. Define

$$\boxed{\langle \alpha, \beta \rangle = \int_M \alpha \wedge * \beta. \quad = \langle \beta, \alpha \rangle}$$

Then if g is pos. def. (a Riemannian manifold) then this scalar product is also pos. def., i.e., $\langle \alpha, \alpha \rangle \geq 0$ and $\langle \alpha, \alpha \rangle = 0$ iff $\alpha = 0$.

Notice that if α, β are 0-forms (call them f_1, f_2), then we get the obvious scalar product of them,

$$\langle f_1, f_2 \rangle = \int \Omega \, f_1 f_2 = \int d^m x \sqrt{|g|} \; f_1 f_2 .$$

More generally, if g is pos. def., we have a scalar product on $\Omega^r(M)$ that allows us to define a <u>Hilbert space of r-forms</u>. The functional analysis of this is easiest in the case of compact M.

An example of this scalar product. Let F be the EM field tensor in 4D space time, (maybe curved),

$$F = \frac{1}{2} F_{\mu\nu} \, \theta^\mu \wedge \theta^\nu .$$

Then

$$\langle F, F \rangle = \int F \wedge * F = \frac{1}{2} \int F_{\mu\nu} F^{\mu\nu} \sqrt{|g|} \, d^4 x .$$

This is $2\times$ the EM field action,

$$S_{EM} = \frac{1}{2} \langle F, F \rangle .$$

(But the scalar product is not pos. def. on space-time.)

Now consider interaction of $*$ with exterior deriv. d. Let $\alpha \in \Omega^r(M)$, $\beta \in \Omega^{r-1}(M)$. Then $\langle \alpha, d\beta \rangle$ is meaningful. We define the operator d^+ (the adjoint of d) by

$$\langle \alpha, d\beta \rangle = \langle d^+ \alpha, \beta \rangle , \qquad \forall \, \alpha \in \Omega^r(M), \; \beta \in \Omega^{r-1}(M).$$

d^+ is the unique operator that makes this equation true. Note that

$$d^+ : \Omega^r(M) \to \Omega^{r-1}(M)$$
$$d : \Omega^{r-1}(M) \to \Omega^r(M)$$

(d and d^+ work in opposite directions).

We can find an expression for d^+ as follows.

$$\langle d^+\alpha, \beta\rangle = \langle \alpha, d\beta\rangle = \langle d\beta, \alpha\rangle = \int_M d\beta \wedge * \alpha.$$

But $d(\beta \wedge * \alpha) = d\beta \wedge * \alpha + (-1)^{r-1} \beta \wedge d*\alpha$, so

$$= \int_M d(\beta \wedge * \alpha) - (-1)^{r-1} \int_M \beta \wedge d * \alpha.$$

First term vanishes by Stokes' thm (we assume $\partial M = 0$), so

$$= (-1)^r \int_M \beta \wedge d * \alpha = (-1)^r \int_M \beta \wedge *(*^{-1} d * \alpha)$$

$$= (-1)^r \langle \beta, *^{-1} d * \alpha\rangle = (-1)^r \langle *^{-1} d * \alpha, \beta\rangle.$$

This implies,

$$\boxed{d^+ = (-1)^r *^{-1} d *} \quad \text{acting on } r\text{-forms.}$$

In this expression, $*^{-1}$ acts on an $(m-r+1)$-form, so

$$*^{-1} = \operatorname{sgn}(g) (-1)^{(m-r+1)(r-1)} *.$$

Since

$$r + (m-r+1)(r-1) \equiv mr + m + 1 \pmod 2,$$

we have

$$\boxed{d^+ = \operatorname{sgn}(g)(-1)^{mr+m+1} * d *} \quad \text{alternative expression, acting on } r\text{-forms.}$$

We note the identity,

$$\boxed{d^+d^+ = 0}$$

which is easily proved,

$$d^+d^+ = *d**d* = \underset{\text{sign of } **}{\pm} *dd* = 0.$$

Note that d^+ annihilates any 0-form,

$$d^+ f = 0, \qquad f \in \mathcal{F}(M)$$

because there are no (−1)-forms.

Summary:

$$\Omega = \sqrt{|g|}\;\theta^1 \wedge \dots \wedge \theta^m = \frac{\sqrt{|g|}}{m!}\,\varepsilon_{\mu_1 \dots \mu_m}\,\theta^{\mu_1} \wedge \dots \wedge \theta^{\mu_m}$$

$$\Omega_{\mu_1 \dots \mu_m} = \sqrt{|g|}\;\varepsilon_{\mu_1 \dots \mu_m}$$

$$\Omega^{\mu_1 \dots \mu_m} = \frac{\operatorname{sgn}(g)}{\sqrt{|g|}}\,\varepsilon_{\mu_1 \dots \mu_m}$$

$$\Omega = *1$$

$$* : \Omega^r(M) \to \Omega^{m-r}(M)$$

$$*\left(\theta^{\mu_1} \wedge \dots \wedge \theta^{\mu_r}\right) = \frac{1}{(m-r)!}\,\Omega^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_{m-r}}\left(\theta^{\nu_1} \wedge \dots \wedge \theta^{\nu_{m-r}}\right)$$

$$\left.\begin{aligned} ** &= \operatorname{sgn}(g)(-1)^{r(m-r)} \\ *^{-1} &= \operatorname{sgn}(g)(-1)^{r(m-r)}\,* \end{aligned}\right\} \text{ on } \omega \in \Omega^r(M)$$

$$\alpha \wedge *\beta = \left(\frac{1}{r!}\,\alpha_{\mu_1 \dots \mu_r}\,\beta^{\mu_1 \dots \mu_r}\right)\Omega, \qquad \alpha, \beta \in \Omega^r(M)$$

$$\alpha \wedge *\beta = \beta \wedge *\alpha$$

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge *\beta = \langle \beta, \alpha \rangle \qquad (\text{pos def. if } g \text{ pos def.})$$

$$\langle \alpha, d\beta \rangle = \langle d^\dagger \alpha, \beta \rangle, \qquad \forall\ \alpha \in \Omega^r(M),\ \beta \in \Omega^{r-1}(M).$$

$$d^\dagger : \Omega^r(M) \to \Omega^{r-1}(M)$$

$$d : \Omega^r(M) \to \Omega^{r+1}(M)$$

$$d^\dagger = (-1)^r\, *^{-1} d\, * = \operatorname{sgn}(g)(-1)^{mr+m+1} * d * . \quad (\text{on } \omega \in \Omega^r(M))$$

$$d^\dagger d^\dagger = 0$$

Now work out the action of d^+ on a 1-form $\alpha \in \Omega^1(M)$. (We know d^+ annihilates 0-forms). $d^+\alpha$ is a scalar, want to find it. Write $\alpha = \alpha_\mu \theta^\mu$.

First compute $*\alpha$:

$$*\alpha = \frac{\alpha_\mu}{(m-1)!} \, \Omega^\mu{}_{\nu_2 \ldots \nu_m} \, (\theta^{\nu_2} \wedge \ldots \wedge \theta^{\nu_m}) \qquad \text{(Raise + lower } \mu\text{)}$$

$$= \frac{1}{(m-1)!} \, \alpha^\mu \, \underbrace{\boxed{\Omega_{\mu\nu_2 \ldots \nu_m}}}_{\sqrt{|g|}\, \varepsilon_{\mu\nu_2 \ldots \nu_m}} \, (\theta^{\nu_2} \wedge \ldots \wedge \theta^{\nu_m})$$

$$d*\alpha = \frac{1}{(m-1)!} \left(\sqrt{|g|}\, \alpha^\mu\right)_{,\sigma} \, \varepsilon_{\mu\nu_2 \ldots \nu_m} \, \underbrace{\theta^\sigma \wedge \theta^{\nu_2} \wedge \ldots \wedge \theta^{\nu_m}}_{= \varepsilon_{\sigma\nu_2 \ldots \nu_m} \underbrace{\theta^1 \wedge \ldots \wedge \theta^m}_{\frac{\Omega}{\sqrt{|g|}}}}$$

$$d*\alpha = \frac{1}{(m-1)!} \, \frac{1}{\sqrt{|g|}} \left(\sqrt{|g|}\, \alpha^\mu\right)_{,\sigma} (m-1)! \, \operatorname{sgn}\begin{pmatrix}\sigma \\ \mu\end{pmatrix} \Omega$$

Note: $\operatorname{sgn}\begin{pmatrix}\sigma \\ \mu\end{pmatrix} = \delta^\sigma_\mu$. More generally,

$$\operatorname{sgn}\begin{pmatrix}\sigma_1 \ldots \sigma_r \\ \mu_1 \ldots \mu_r\end{pmatrix} = \begin{vmatrix} \delta^{\sigma_1}_{\mu_1} & \ldots & \delta^{\sigma_1}_{\mu_r} \\ \vdots & & \vdots \\ \delta^{\sigma_r}_{\mu_1} & \ldots & \delta^{\sigma_r}_{\mu_r} \end{vmatrix}$$

$$= \frac{1}{\sqrt{|g|}} \left(\sqrt{|g|}\, \alpha^\mu\right)_{,\mu} \Omega .$$

A digression on a useful theorem. Let $X = X^\mu e_\mu$ be a vector field. Then

We assume Levi-Civita connection

$$\boxed{X^\mu{}_{;\mu} = \frac{1}{\sqrt{|g|}}\left(\sqrt{|g|}\, X^\mu\right)_{,\mu}}$$

useful formula.

We used this theorem in computing Einstein field eqns. from Lagrangian.

To prove it, expand RHS,

$$\text{RHS} = \frac{1}{\sqrt{|g|}}\left(\sqrt{|g|}\right)_{,\mu} X^\mu + X^\mu{}_{,\mu}$$

But by the formula for the derivative of a determinant,

$$= \frac{1}{2}\left(g^{\alpha\beta} g_{\alpha\beta,\mu}\right) X^\mu + X^\mu{}_{,\mu}.$$

Now
$$X^\mu{}_{;\nu} = X^\mu{}_{,\nu} + \Gamma^\mu_{\alpha\nu} X^\alpha,$$

so
$$\text{LHS} = X^\mu{}_{;\mu} = X^\mu{}_{,\mu} + \Gamma^\mu_{\alpha\mu} X^\alpha.$$

Also,
$$\Gamma^\mu_{\alpha\beta} = \tfrac{1}{2} g^{\mu\nu}\left(g_{\nu\alpha,\beta} + g_{\nu\beta,\alpha} - g_{\alpha\beta,\nu}\right),$$

so
$$\Gamma^\mu_{\alpha\mu} = \tfrac{1}{2} g^{\mu\nu}\left(\cancel{g_{\nu\alpha,\mu}} + g_{\nu\mu,\alpha} - \cancel{g_{\alpha\mu,\nu}}\right),$$

(two terms cancel by exchange $\mu \leftrightarrow \nu$ and symmetry)

$$\Gamma^\mu_{\alpha\mu} = \tfrac{1}{2} g^{\mu\nu} g_{\mu\nu,\alpha}.$$

So

$$\text{LHS} = X^\mu{}_{,\mu} + \tfrac{1}{2} g^{\mu\nu} g_{\mu\nu,\alpha} X^\alpha = \text{RHS}. \qquad \text{QED}$$

So to go back, we have

$$d{*}\alpha = \frac{1}{\sqrt{|g|}}\left(\sqrt{|g|}\,\alpha^{\mu}\right)_{,\mu}\Omega \quad = \quad \alpha^{\mu}{}_{;\mu}\,\Omega$$

Now apply $(-1)^r {*}^{-1} = -{*}^{-1}$, note that $\Omega = {*}1$ so ${*}^{-1}\Omega = 1$, get

$$d^{+}\alpha = -{*}^{-1} d{*}\alpha = -\alpha^{\mu}{}_{;\mu}.$$

$$\boxed{d^{+}\alpha = -\alpha^{\mu}{}_{;\mu}}$$

minus

It is the covariant ~~do~~ "divergence" of α (converted to a vector via g).

Note: in special case $\alpha = df$ $\quad (f \in \Omega^0(M))$,

we have

$$d^{+}df = -f^{;\mu}{}_{;\mu}.$$

This is (minus) the obvious generalization of the Laplacian to curved spaces,

$$-\nabla^2 f = -f_{,ii} \quad \text{on Euclidean } \mathbb{R}^n.$$

Another note on this result:

$$\int_M d^n x\,\sqrt{|g|}\;\alpha^{\mu}{}_{;\mu} = -\int_M (d^{+}\alpha)\,\Omega = -\int_M d^{+}\alpha \wedge {*}1$$

$$= -\langle d^{+}\alpha, 1\rangle = -\langle \alpha, d1\rangle = 0.$$

A more straightforward way to see the same thing is to use integration by parts,

$$\int d^m x \sqrt{|g|}\ \alpha^\mu{}_{;\mu} = \int d^m x \ (\sqrt{|g|}\,\alpha^\mu)_{,\mu} = 0.$$

You have to convert a covariant deriv. to an ordinary deriv. if you want to integrate by parts.

Now Hodge $*$ and Maxwell eqns (E+M). Already noted,

$$S_{EM} = \langle F, F \rangle = \int F \wedge * F.$$

~~The~~ Maxwell eqns in SR:

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} \qquad \text{or} \qquad F = dA, \qquad F = \tfrac{1}{2} F_{\mu\nu}\, dx^\mu \wedge dx^\nu$$

$$F_{[\mu\nu,\sigma]} = 0 \qquad \text{or} \qquad dF = 0$$

$$F^{\mu\nu}{}_{,\nu} = J^\mu$$

$$J^\mu{}_{,\mu} = 0$$

We use the comma goes to semicolon rule to put these into GR. For example,

$$F_{[\mu\nu;\sigma]} = 0.$$

Question: does this still mean $dF = 0$? (There are extra terms involving Γ from the covariant derivatives). Answer: Yes, because in the LC connection, all the Γ terms cancel when computing the components of an exterior derivative of any ~~scalar valued~~ (real-valued, i.e., not Lie algebra-valued) form. Thus,

$$F_{[\mu\nu;\sigma]} = 0 \Rightarrow F_{[\mu\nu,\sigma]} = 0 \Rightarrow dF = 0 \Rightarrow$$

$$\Rightarrow F = dA \ \text{(Poincaré lemma)} \Rightarrow F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$$

$$\Rightarrow F = A_{\nu;\mu} - A_{\mu;\nu}.$$

charge conservation

As for $J^{\mu}{}_{,\mu} = 0$ (in SR) it becomes $J^{\mu}{}_{;\mu} = 0$ (in GR)

Now define

$$J = J_{\mu}\, dx^{\mu} \qquad \text{(current 1-form)},$$

and charge conservation becomes

$$d^{+} J = 0.$$

Finally, as for $F^{\mu\nu}{}_{,\nu} = J^{\mu}$ (in SR), it becomes $F^{\mu\nu}{}_{;\nu} = J^{\mu}$ (in GR). It can be shown (exercise for you) that this is equivalent to

$$d^{+} F = J,$$

which is consistent with charge conservation because $d^{+}J = d^{+}d^{+}F = 0$.

Summary of Maxwell equns:

$$\boxed{\begin{array}{ll} F = dA, & d^{+}J = 0. \\ dF = 0 & \\ d^{+}F = J & \end{array}}$$

We can use these equns to get a wave equn. for A:

$$d^{+}F = d^{+}dA = J.$$

Although $d^{+}d$ acting on scalars (we saw above) is (minus) the covariant Laplacian (i.e., d'Alembertian in space-time), this is not quite true for forms of higher rank ($r \geq 1$). For arb. forms we define,

$$\Delta = d^{+}d + dd^{+}.$$

This agrees with the case of scalars since $d^{+}f = 0$ (any scalar f),

so

$$\Delta f = d^{+}df.$$

But on a 1-form such as A we have

$$\Delta A = J - dd^{+}A.$$

The term on the RHS vanishes if we choose Lorentz gauge, $d^{+}A = 0$.

[Think: $-\nabla^2 \vec{A} = \vec{J} - \nabla(\nabla\cdot\vec{A})$ in NR magnetostatics.]

Now we explore the properties of the operator Δ.

$$\boxed{\begin{aligned}\Delta &= d^{+}d + dd^{+} \\ &= (d+d^{+})^2\end{aligned}}$$

(defn)

since $d^2 = d^{+2} = 0$.

Actually, to simplify the functional analysis it helps to assume that M is also <u>compact</u>.

In the following we assume the <u>Riemannian case</u>, so g is <u>positive def.</u>

This means that $\langle\,,\rangle$ is also positive def., so that

$$\langle\alpha,\alpha\rangle \geq 0, \quad \text{and} \quad \langle\alpha,\alpha\rangle = 0 \text{ iff } \alpha = 0. \qquad (\text{any form } \alpha).$$

First note that Δ is ~~po~~ Hermitian,

$$\begin{aligned}\langle\alpha, \Delta\beta\rangle &= \langle\alpha, d^{+}d\beta\rangle + \langle\alpha, dd^{+}\beta\rangle \\ &= \langle d\alpha, d\beta\rangle + \langle d^{+}\alpha, d^{+}\beta\rangle \\ &= \langle d^{+}d\alpha, \beta\rangle + \langle dd^{+}\alpha, \beta\rangle \\ &= \langle\Delta\alpha, \beta\rangle.\end{aligned}$$

(More simply, just use the rules of $+$ on products of operators, as in QM).

Next note that Δ is a ~~positive definite~~ <u>nonnegative definite operator</u>,

i.e., $\langle\alpha, \Delta\alpha\rangle \geq 0 \quad \forall\, \alpha$. Proof is easy: