Now we turn to Riemannian manifolds. These are manifolds that posses a <u>metric tensor</u>. We have already discussed metric tensors on a vector space V; now we promote this idea into a field, by identifying the former V with $T_x M$ (one metric in each tangent space). The result is a type $(0,2)$ tensor g,

$$g|_x : T_x M \times T_x M \to \mathbb{R} \qquad \text{(at a point)}$$

$$g : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{F}(M) \qquad \text{(as a field)}.$$

At each point $x \in M$, g satisfies the requirements of a metric:

(1) $\quad g(X,Y) = g(Y,X) \qquad$ (symmetric)

(2) $\quad g$ is nonsingular.

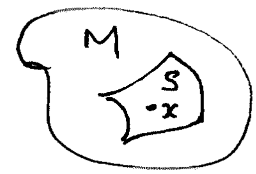
Property (2) can be expressed in terms of the component matrix $g_{\mu\nu}$ of g w.r.t. some basis $\{e_\mu\}$,

(2′) $\quad \det(g_{\mu\nu}) \neq 0,$

a stmt independent of basis. By an orthogonal change of basis, $g_{\mu\nu}$ can be diagonalized. The eigenvalues are not invariant under change of basis (which generally need not be orthogonal), but their signs are. In fact, by scaling the basis vectors by a factor $a \neq 0$, after $g_{\mu\nu}$ has been diagonalized, the eigenvalues are scaled by $a^2 > 0$. Thus they can be scaled to ± 1 (0 is excluded since $g_{\mu\nu}$ is nonsingular). The number of $+1$'s and -1's in this final form is an invariant property of $g_{\mu\nu}$. ~~all~~ The list of these numbers is the <u>signature</u> of g.

If the signature contains only $+1$'s, then $g_{\mu\nu}$ is positive definite, and M is said to be a <u>Riemannian manifold</u>. If some are $+1$ and others -1, then M is <u>pseudo-Riemannian</u>. The signature of g cannot change as we move around on M because eigenvalues are not allowed to pass through 0. (assuming M is connected.)

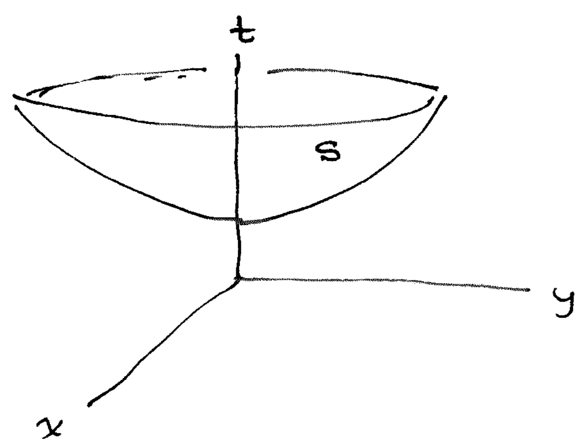
Let S be a submanifold of M, a (pseudo)-Riemannian manifold. Then g (on M) can be restricted to S, creating a type $(0,2)$ tensor on S. (In fact, any purely covariant tensor, a 2-form for example, can be restricted to submanifolds in the same manner.) We just define



$$g|_S(X,Y) = g(X,Y),$$

where $X, Y \in T_xS$, are reinterpreted as elements of T_xM. $g|_S$ is then a tensor field on S. If g is positive definite (Riemannian case), then $g|_S$ is also, and S becomes a Riemannian manifold (every submanifold is Riemannian). But if M is pseudo-Riemannian, then $g|_S$ ~~does~~ is not necessarily nonsingular everywhere.

<u>Example:</u> Let $M = \mathbb{R}^4$ with Minkowski metric, let S = unit hyperbola (or mass shell):



Metric on M:

$$-dt^2 + dx^2 + dy^2 + dz^2$$

Metric on S: turns out to be Riemannian (pos. def.) but not flat, S is surface of const. negative curvature (Lobachevskian plane).

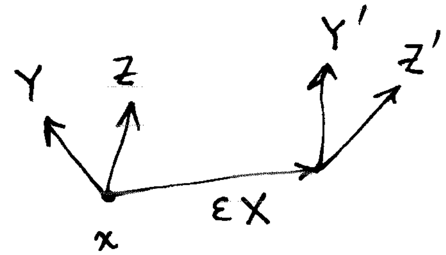
Let x^μ be coordinates. Then in the coordinate basis,

g is

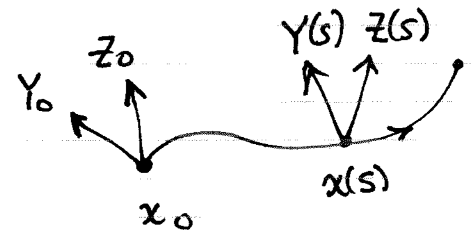
$$g = g_{\mu\nu}\, dx^\mu \otimes dx^\nu,$$ often written without the $\otimes$. for short.

Now suppose the manifold has both a connection and a metric. We emphasize that a <u>metric</u> and a <u>connection</u> are two different geometrical constructions. You can have a manifold with a connection but without a metric. However if you do have both, you can compute ∇g.

A ~~metric~~ connection for which $\nabla_X g = 0$ (for all X) is called a <u>metric connection</u>. If you have a metric connection, then the scalar product of parallel transported vectors is ~~preserved~~ constant: Let $Y, Z \in T_x M$ be two vectors parallel transported along εX to a new point $x + \varepsilon X$, to give new vectors Y', Z':



Then $g(Y,Z) = g(Y',Z')$, if you use a metric connection. Similarly, if $Y(s)$, $Z(s)$ are the parallel transports of $Y_0, Z_0 \in T_{x_0} M$ along a curve,



then

$$\frac{d}{ds}\left[Y(s)_\mu \, Z(s)^\mu\right] = 0.$$

The condition $\nabla_X g = 0$, $\forall\, X$, implies:

$$g_{\mu\nu,\alpha} = \Gamma^\beta_{\alpha\mu}\, g_{\beta\nu} + \Gamma^\beta_{\alpha\nu}\, g_{\mu\beta}.$$

This eqn. can be solved ~~in terms~~ for Γ in terms of g and its derivatives and the torsion tensor. First define

$$\Gamma_{\mu\alpha\beta} = g_{\mu\nu}\, \Gamma^\nu_{\alpha\beta}.$$

Then write

$$S_{\mu\alpha\beta} = \Gamma_{\mu\alpha\beta} + \Gamma_{\mu\beta\alpha}$$

$$T_{\mu\alpha\beta} = \Gamma_{\mu\alpha\beta} - \Gamma_{\mu\beta\alpha}$$

these are the symmetric and antisymmetric parts of Γ. The anti-symmetric part is the same as the torsion tensor (but the symmetric part is not a tensor). So we have

$$\Gamma_{\mu\alpha\beta} = \frac{1}{2}\left(S_{\mu\alpha\beta} + T_{\mu\alpha\beta}\right)$$

$$\begin{aligned} g_{\mu\nu,\alpha} &= \Gamma_{\nu\alpha\mu} + \Gamma_{\mu\alpha\nu} \\ &= \frac{1}{2}\left(S_{\nu\alpha\mu} + S_{\mu\alpha\nu} + T_{\nu\alpha\mu} + T_{\mu\alpha\nu}\right) \end{aligned}$$

$$g_{\nu\alpha,\mu} = \frac{1}{2}\left(S_{\alpha\mu\nu} + S_{\nu\mu\alpha} + T_{\alpha\mu\nu} + T_{\nu\mu\alpha}\right)$$

$$g_{\alpha\mu,\nu} = \frac{1}{2}\left(S_{\mu\nu\alpha} + S_{\alpha\nu\mu} + T_{\mu\nu\alpha} + T_{\alpha\nu\mu}\right)$$

Solve for $S_{\alpha\mu\nu} = S_{\alpha\nu\mu}$

$$g_{\alpha\mu,\nu} + g_{\nu\alpha,\mu} - g_{\mu\nu,\alpha} = \underbrace{S_{\alpha\mu\nu}}_{\to\, 2\Gamma_{\alpha\mu\nu} - T_{\alpha\mu\nu}} + T_{\mu\nu\alpha} + T_{\nu\mu\alpha},$$

So,

$$\Gamma^{\beta}{}_{\mu\nu} = \underbrace{\frac{1}{2} g^{\alpha\beta}\left(g_{\alpha\mu,\nu} + g_{\nu\alpha,\mu} - g_{\mu\nu,\alpha}\right)} + \frac{1}{2}\left(T^{\beta}{}_{\mu\nu} + T_{\mu}{}^{\beta}{}_{\nu} + T_{\nu}{}^{\beta}{}_{\mu}\right)$$

$\to$ denoted $\left\{ {\beta \atop \mu\nu} \right\}$ = Christoffel symbols.

As claimed, we have Γ in terms of g and the torsion, for a metric connection. If the torsion vanishes, then

$$\Gamma^{\mu}_{\alpha\beta} = \left\{ {\mu \atop \alpha\beta} \right\} = \Gamma^{\mu}_{\beta\alpha}.$$

The connection that satisfies this is a special metric connection, called the Levi-Civita connection. In a sense it is the simplest metric connection.

Under the Levi-Civita connection, a connection geodesic is the same as a metric geodesic, i.e.,

$$0 = \delta \int ds = \delta \int \sqrt{g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds}}\, ds$$

$$\Rightarrow \quad \frac{d^2 x^\mu}{ds^2} + \left\{ {\mu \atop \alpha\beta} \right\} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = 0.$$

Under a metric connection, the parallel transport of vectors around a loop preserves scalar products, so the holonomy is an orthogonal transformation $O(n)$ for a Riemannian manifold, or an element of $O(r,s)$ for a pseudo-Riemannian manifold. For example, in GR, the holonomy is an element of the Lorentz group $O(3,1)$ of special relativity.

Under a metric connection, the curvature tensor has further symmetries. With g we can raise and lower indices, and talk about $R_{\mu\nu\alpha\beta}$. The $\mu\nu$ indices now refer to an element of the Lie algebra of an orthogonal group, so they are antisymmetric, ie

$$R_{\mu\nu\alpha\beta} = - R_{\nu\mu\alpha\beta}.$$

This only requires $\nabla g = 0$, not $T = 0$. But...

Finally, if we assume both a metric connection $\nabla g = 0$ and vanishing torsion $T=0$, i.e., the Levi-Civita connection, then there is another symmetry,

$$R_{\mu\nu\alpha\beta} = R_{\alpha\beta\mu\nu}.$$

You can compute the number of independent components of $R^{\mu}{}_{\nu\alpha\beta}$ under various assumption. For example, if you just have a connection and nothing else there are

$$\frac{m^3(m-1)}{2} \qquad m = \dim M.$$

indep. components (only symmetry is 2-form symmetry). But with the Levi-Civita connection (do the combinatorics) you find

$$\frac{m^2(m^2-1)}{12}.$$

indep. components. Table:

m	$\frac{m^2(m^2-1)}{12}$
0	0
1	0
2	1
3	6
4	20

Easy to understand why only one component for $m=2$. A 2-form on a 2D space has only one indep. components, and $SO(2)$ is 1-dimensional. In this case, the holonomy around any loop (infinitesimal or otherwise) is specified by an angle of rotation θ, and R can be reduced to a scalar-valued 2-form (the usual kind). Thus, it represents a kind of "angle density" on M, the integral of this over some area gives the holonomy on parallel transporting around the boundary. For example, on the usual 2-sphere with the obvious metric and connection, you find

$$R = \sin\theta \, d\theta \wedge d\phi = \text{"} d\Omega \text{"}$$

the solid angle.

In higher dimensions the holonomy group is usually non abelian, so you can't get the ~~angle of~~ holonomy on going around a finite loop by integrating the 2-form (R) over the interior.

Some tensors important in GR. → assume LC connection. By contracting the Riemann tensor, we get the Ricci tensor, which can be contracted to the curvature scalar:

Ricci: $$R_{\nu\beta} = R^{\mu}{}_{\nu\mu\beta}$$

Curv. Scalar: $$R = R^{\nu}{}_{\cdot\nu} = R^{\mu\nu}{}_{\cdot\cdot\mu\nu}.$$

The 2nd Bianchi identity implies a differential equation satisfied by these:

$$R^{\mu}{}_{\nu\alpha\beta;\gamma} + R^{\mu}{}_{\cdot\nu\gamma\alpha;\beta} + R^{\mu}{}_{\cdot\nu\beta\gamma;\alpha} = 0 \qquad (\text{contract } \mu, \alpha)$$

$$R_{\nu\beta;\gamma} + \underbrace{R^{\mu}{}_{\nu\gamma\mu;\beta}}_{= -R^{\mu}{}_{\nu\mu\gamma;\beta} = -R_{\nu\gamma;\beta}} + R^{\mu}{}_{\cdot\nu\beta\gamma;\mu} = 0$$

$$R_{\nu\beta;\gamma} - R_{\nu\gamma;\beta} + R^{\mu}{}_{\cdot\nu\beta\gamma;\mu} = 0. \qquad (\text{contract } \nu, \beta)$$

$$R_{;\gamma} - R^{\nu}{}_{\gamma;\nu} + \underbrace{R^{\mu\nu}{}_{\nu\gamma;\mu}}_{= -R^{\nu\mu}{}_{\nu\gamma;\mu} = -R^{\mu}{}_{\gamma;\mu}.} = 0$$

$$\Rightarrow \quad R^{\mu}{}_{\nu;\mu} - \frac{1}{2} R_{;\nu} = 0.$$

shows appearance of Einstein tensor,

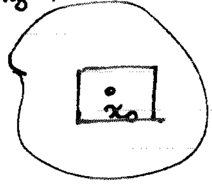
stress-energy, not torsion

$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R$$

satisfies $G^{\mu}{}_{\nu;\mu} = 0$, necessary for field equations $G_{\mu\nu} = 8\pi T_{\mu\nu}$

since $T^{\mu}{}_{\nu;\mu} = 0$ (local energy-momentum conservation).

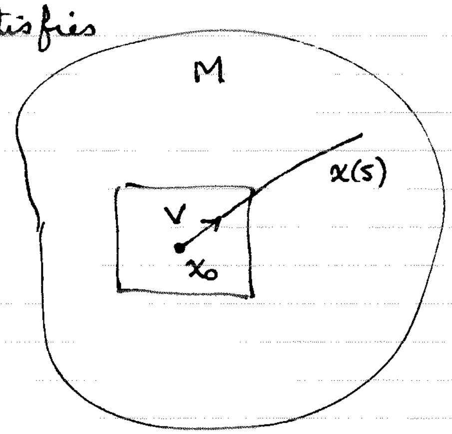
Now we consider Riemann normal coordinates. These are coordinates that simplify the expressions for tensors and covariant derivatives as much as possible in a neighborhood of a given point. We know that a small piece of M is approximately flat. Riemann normal coordinates take advantage of this to make various expression look as much as possible like those on a flat space. The idea is the following. The tangent space $T_{x_0}M$ looks like a small piece of M in the neighborhood of x_0. We can impose linear coordinates on $T_{x_0}M$, which is a vector space. Can those coordinates somehow be extended to make coordinates on M itself?



Let $V \in T_{x_0}M$ be a vector in the "initial" tangent space (at x_0), and consider the geodesic $x(s)$ that satisfies

$$x(0) = x_0$$

$$\frac{dx}{ds}(0) = V.$$



Let the point reached after elapsed parameter s be $p(V, s)$.

The equation of the geodesic is

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta}(x)\,\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0.$$

Here x^μ = any coordinates in a neighborhood of x_0.

This equation is homogeneous in s, so if $x^\mu(s)$ is a solution, then so is $x^\mu(ks)$ for $k \in \mathbb{R}$. But they satisfy different initial conditions,

if $\left.\dfrac{dx^\mu(s)}{ds}\right|_{s=0} = V^\mu$, then $\left.\dfrac{d}{ds}\right|_{s=0} x^\mu(ks) = kV^\mu$.

In other words, if you scale the initial vector by k, then the curve is traversed k times as fast. In other words,

$$p(kV,\ s/k) = p(V,s).$$

Thus $p(V,s) = p(sV,1)$. The point $p(V,s)$ actually depends only on the product sV. Thus we can define,

$$\exp : T_{x_0}M \to M : V \mapsto p(V,1).$$

This map is onto in some neighborhood of x_0, which is "obvious" if you think of $T_{x_0}M$ being a small piece of M (for small vectors in $T_{x_0}M$).

Now choose a basis in $T_{x_0}M$ (could be $\partial/\partial x^\mu|_{x_0}$), with respect to which V has coordinates V^μ. Then use exp to map these coordinates on $T_{x_0}M$ onto M itself. But change the symbol to w^μ, to avoid confusion with coords V^μ on $T_{x_0}M$. Then the geodesics are just <u>radial lines</u> in the w^μ-coordinates,

$$w^\mu(t) = t\,\xi^\mu \qquad \text{(eqn. of geodesic in } w^\mu \text{ coords).}$$

The coordinates w^μ are called <u>Riemann normal coordinates</u>.

Various tensor fields simplify in RNC. Consider first Γ. The eqn. of a geodesic in RNC is

$$\frac{d^2 w^\mu}{dt^2} + \Gamma^\mu_{\alpha\beta}(t\xi)\frac{dw^\alpha}{dt}\frac{dw^\beta}{dt} = 0$$

$$\text{or} \quad \Gamma^\mu_{\alpha\beta}(t\xi)\,\xi^\alpha\xi^\beta = 0.$$

setting $t=0$ gives $\Gamma^\mu_{\alpha\beta}(0)\,\xi^\alpha\xi^\beta = 0.$

This holds for all ξ, and since $\Gamma^\mu_{\alpha\beta}$ is symmetric in $(\alpha\beta)$ (Levi-Civita connection), it follows that

$$\Gamma^\mu_{\alpha\beta}(0) = 0 \quad \text{in RNC.}$$

From this it follows that

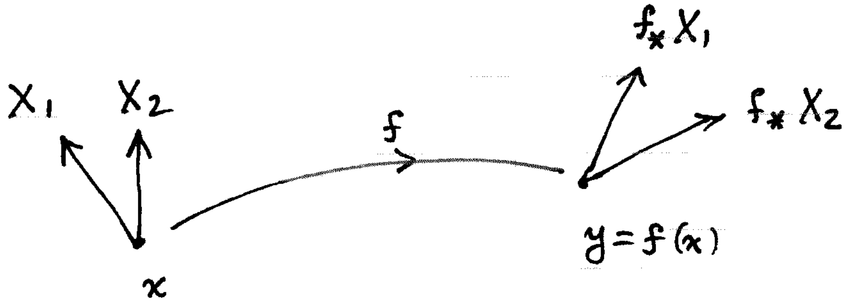
$$g_{\mu\nu,\alpha}(0) = 0 \qquad \text{since } \nabla g = 0.$$

So if you expand the metric tensor in RNC about x_0, you find only 2nd order corrections. In general the components $g_{\mu\nu}$ cannot be constant (that would imply a flat space), but in the right coordinates (namely, RNC) they can be made constant through first order terms in the displacement. The Riemann tensor does not vanish at x_0 (it cannot, in general, since it is a tensor), but the expression in terms of the metric simplifies since $\Gamma^\mu_{\alpha\beta}(0) = 0$. You find

$$R^\mu{}_{\nu\alpha\beta}(0) = \Gamma^\mu_{\beta\nu,\alpha}(0) - \Gamma^\mu_{\alpha\nu,\beta}(0).$$

The vanishing of the Riemann tensor is the integrability condition that there should exist a coordinate system in which $g_{\mu\nu} = \text{const}$. This would be a flat space.

Now for some remarks about conformal transformations and isometries. Let (M, g) be a (pseudo)-Riemannian manifold, with Levi-Civita connection ∇, ~~can~~ and consider a map $f: M \to M$. [This discussion is easily generalized to the case $f: M \to N$, between two Riem. manifolds.]. Let $y = f(x)$, some $x \in M$, and let $X_1, X_2 \in T_x M$.



We compare the scalar products $g|_x (X_1, X_2)$ and $g|_{f(x)} (f_* X_1, f_* X_2)$. If these are proportional by some positive scale factor, written $e^{2\sigma(x)}$ where $\sigma: M \to \mathbb{R}$ is a scalar field, i.e., if $\exists\, \sigma$ such that

$$g|_{f(x)} (f_* X_1, f_* X_2) = e^{2\sigma(x)} g|_x (X_1, X_2)$$

for all $X_1, X_2 \in T_x M$, and all $x \in M$, then we say $f: M \to M$ is a <u>conformal transformation</u>. The condition can be written more compactly as

pull back, superscript *

$$\boxed{f^* g = e^{2\sigma} g.} \qquad \text{(Defn of conformal transf. } f\text{)}.$$

If this equation holds for $\sigma = 0$, i.e., if

$$\boxed{f^* g = g} \qquad \text{(Defn. of isometry } f\text{)}.$$

Then f is said to be an <u>isometry</u>. An isometry is a special case of a conformal transformation. Under conformal transformation, scalar products are preserved up to a scaling; this preserves angles but not necessarily lengths. Under an isometry, both lengths and angles are preserved.

Historical note: Conformal transformations entered physics with Weyl's 1919 attempt to unify E+M and general relativity (then very new). In Weyl's theory, the integral $\int A_\mu dx^\mu$ of the E+M vector potential was interpreted as a scale factor for a scaling of the metric. The idea failed, however, because this scale factor is path dependent. But this is where the word "gauge" comes from in "gauge transformation." Later $\int A_\mu dx^\mu$ was reinterpreted as part of the phase of the quantum wave function, the change of which is still called a gauge transformation.

In modern times conformal field theories are important as exactly solvable 2D models of quantum field theories, in 2D critical phenomena, and in string theory.

An example of a conformal transformation in 2D is any analytic complex function $w = w(z)$ on the complex plane. Let $z = x+iy$, $w = u+iv$. Then can easily show,

$$dx^2+dy^2 = \frac{du^2+dv^2}{u_x^2+u_y^2}, \qquad u_x = \frac{\partial u}{\partial x}, \quad u_y = \frac{\partial u}{\partial y}$$

because of Cauchy-Riemann conditions. Thus the mapping $z \mapsto w$ is conformal.

Examples of isometries are translations and rotations (the Euclidean group) on Euclidean $\mathbb{R}^n$.

A concept closely related to conformal transformations is the following. Let M be a manifold with two metrics g and $\bar{g}$, and suppose

$$\bar{g} = e^{2\sigma} g.$$

Then g and $\bar{g}$ are said to be <u>conformally related</u>. Let δ be a metric that in some coordinates has the form $g_{\mu\nu} = \delta_{\mu\nu}$ (flat space). Then if $\bar{g} = e^{2\sigma}\delta$, then $\bar{g}$ is said to be <u>conformally flat</u>.

As discussed, the integrability condition for the existence of a coordinate system such that $g_{\mu\nu} = \delta_{\mu\nu}$ is the vanishing of the Riemann tensor, $R^{\mu}{}_{\nu\alpha\beta} = 0$. It turns out that there is another (less restrictive) condition that g satisfies if it is conformally flat, i.e., if there exists a coord. system and scalar field σ such that $g_{\mu\nu} = e^{2\sigma}\delta_{\mu\nu}$. This condition is the vanishing of the <u>Weyl tensor</u>,

$$W_{\mu\nu\alpha\beta} = R_{\mu\nu\alpha\beta} + \frac{1}{m-2}\left(g_{\mu\beta}R_{\alpha\nu} - g_{\mu\alpha}R_{\beta\nu} + g_{\alpha\nu}R_{\mu\beta} - g_{\beta\nu}R_{\mu\alpha}\right)$$

$$+ \frac{1}{(m-1)(m-2)}\left(g_{\mu\alpha}g_{\beta\nu} - g_{\mu\beta}g_{\alpha\nu}\right)R,$$

where $m = \dim M$, $R_{\mu\nu}$ = Ricci tensor, R = curvature scalar. W has the property that it is invariant under conformal transformations; hence if g is conformally flat, then $W=0$. Conversely, if $W=0$, then for $m \geq 4$, the metric is conformally flat; but for $m=3$ $W=0$ always, and for $m=2$, any g is conformally flat.

Back to isometries. It's easy to show that given (M, g), the set of isometries forms a group. This can be thought of as an abstract group G whose action Φ_a on M is the set of isometries, that is

$$\Phi_a^* g = g \,, \qquad a \in G.$$

(g = metric, not a group element). Assume that G is a Lie group, and let $a = \exp(tV)$ where $V \in \mathfrak{g}$ = Lie algebra of G. Then

$$\left.\frac{d}{dt}\right|_{t=0} \Phi^*_{\exp(tV)} = V_M = \text{induced vector field, call it } X_V \in \mathfrak{X}(M).$$

This eqn. holds if both sides act on scalars. For other tensors, (such as g) replace RHS by $\mathcal{L}_{X_V}$ (the Lie derivative). Thus if G is the isometry group, then

$$\Phi^*_{\exp(tV)}\, g = g \,,$$

or, applying $\left.\frac{d}{dt}\right|_{t=0}$,

$$\mathcal{L}_{X_V} g = 0.$$

A vector field $X \in \mathfrak{X}(M)$ such that $\mathcal{L}_X g = 0$ is called a <u>Killing vector field</u>. Killing vector fields represent infinitesimal isometries. A problem is to find the Killing vector fields given g.

Let X be a Killing vector field. Then by writing $\mathcal{L}_X$ in components, we have

$$(\mathcal{L}_X g)_{\alpha\beta} = X^\mu g_{\alpha\beta,\mu} + X^\mu{}_{,\alpha}\, g_{\mu\beta} + X^\mu{}_{,\beta}\, g_{\alpha\mu} = 0.$$

This is a differential equation that X must satisfy. ~~[illegible] a metric connection, so that~~ Use the Levi-Civita connection, so that

$$g_{\alpha\beta,\mu} = \Gamma^\sigma_{\mu\alpha}\, g_{\sigma\beta} + \Gamma^\sigma_{\mu\beta}\, g_{\alpha\sigma}$$

Thus

$$\cancel{X^\sigma_{,\alpha}\, g_{\sigma\beta} + X^\sigma_{,\beta}\, g_{\alpha\sigma}}$$

$$X^\sigma_{,\alpha}\, g_{\sigma\beta} + X^\sigma_{,\beta}\, g_{\alpha\sigma} + \Gamma^\sigma_{\mu\alpha}\, g_{\sigma\beta} X^\mu + \Gamma^\sigma_{\mu\beta}\, g_{\alpha\sigma} X^\mu$$

$$= g_{\sigma\beta} X^\sigma_{;\alpha} + g_{\alpha\sigma} X^\sigma_{;\beta}$$

$$= \boxed{X_{\beta;\alpha} + X_{\alpha;\beta} = 0}$$

This is Killing's eqn, nice compact form for eqn. that Killing vectors must satisfy.

Here are some <u>examples</u> of Killing vector fields on some spaces. First take Euclidean $\mathbb{R}^n$ in standard coordinates. Then $X_{\beta;\alpha} = X_{\beta,\alpha}$, and Killing's eqn. is

$$X_{\beta,\alpha} + X_{\alpha,\beta} = 0, \quad \text{also } X_\beta = X^\beta \quad \text{since } g_{\mu\nu} = \delta_{\mu\nu}.$$

Expand X_α in a Taylor series:

$$\cancel{X_\alpha = a_\alpha + b_{\alpha\beta} x^\alpha + c_{\alpha\beta\gamma} x^\alpha x^\beta x^\gamma +}$$

$$X_\alpha = a_\alpha + b_{\alpha\beta} x^\beta + C_{\alpha\beta\gamma} x^\beta x^\gamma + \cdots$$

so

$$X_{\alpha,\beta} = b_{\alpha\beta} + 2 C_{\alpha\beta\gamma} x^\gamma + \cdots$$

$$X_{\beta,\alpha} = b_{\beta\alpha} + 2 C_{\beta\alpha\gamma} x^\gamma + \cdots$$

$$0 = (b_{\alpha\beta} + b_{\beta\alpha}) + 2(C_{\alpha\beta\gamma} + C_{\beta\alpha\gamma}) x^\gamma + \cdots$$

Thus $b_{\alpha\beta} = -b_{\beta\alpha}$ (antisymmetric), a_α = any const vector. As for C, it is symmetric in $\beta\gamma$ and antisymm. in $\alpha\beta$, which $\Rightarrow$ $C = 0$. Same for all higher tensors. Thus we find

$$X_\alpha = a_\alpha + b_{\alpha\beta} x^\beta .$$

We ~~may~~ recognize this as an infinitesimal displacement (a_α) composed with an infinitesimal rotation ($b_{\alpha\beta} = -b_{\beta\alpha}$ means $b \in so(n)$). The number of parameters is $m + \frac{m(m-1)}{2} = \frac{m(m+1)}{2}$ ($m = \dim M$). This is the number of lin. indep. Killing vector fields; it is a finite number. In fact, it is the maximum number that a space of m dimensions can have. For this reason, Euclidean $\mathbb{R}^m$ is called a <u>maximally symmetric space</u>. Another example of a Max-Sym. Space is the sphere S^n, ~~with~~ with the induced metric obtained by embedding in Euclidean $\mathbb{R}^{n+1}$. S^n is invariant under $O(n+1)$, a group with $n(n+1)/2$ dimensions. (Same number). Similarly, n-dim. surfaces of const. neg. curvature can be imbedded in Minkowski $\mathbb{R}^{n+1}$, and are maximally symmetric under the $(n+1)$-dimensional Lorentz group.

Since the Killing vectors are the infinitesimal generators of the ~~group~~ action of the isometry group on M, they must form a Lie algebra. This is easy to show directly. Let X, Y be two Killing vector fields, so

$$\mathcal{L}_X g = 0$$
$$\mathcal{L}_Y g = 0 .$$

Then since $[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]}$, we have $\mathcal{L}_{[X,Y]}\, g = 0$, hence $[X,Y]$ is also a Killing vector field.

Can also talk about <u>conformal Killing vectors.</u>

New subject. Now we put the basic equations of metrical geometry into a <u>noncoordinate</u> <u>basis</u>, and also introduce the formalism of Cartan. In 4 dimensions, a noncoordinate basis is sometimes called a <u>tetrad</u> or <u>vierbein</u> (German for "four legs"), because a frame is a set of lin. indep. vectors  a "dreibein". In many dimensions, we may refer to the frame as a <u>vielbein</u> (many legs).

Nakahara distinguishes components w.r.t. to a vielbein from those w.r.t. a coordinate basis by using $\alpha, \beta, \gamma \ldots$ for the vielbein and $\mu, \nu, \lambda, \ldots$ for the coordinate basis. We will just use any indices in the following, but it will be understood that we are working in a non-coordinate basis. (the general case.)

Let $\{e_\mu\}$ be the basis vector fields, assumed to be lin. indep. at each point of some region of space. Let $\{\theta^\mu\}$ be the dual basis of 1-forms, so that

$$\theta^\mu(e_\nu) = \delta^\mu_\nu .$$

The basis vectors satisfy

$$[e_\mu, e_\nu] = c^\sigma_{\mu\nu}\, e_\sigma$$

where $c^\sigma_{\mu\nu} = -c^\sigma_{\nu\mu}$ are the structure constants. (not really const. however). Similarly, we have

$$d\theta^\mu = -\frac{1}{2} c^\mu_{\alpha\beta}\, \theta^\alpha \wedge \theta^\beta .$$

(derived previously). (Here's how:

$$d\theta^\mu(e_\alpha, e_\beta) = e_\alpha[\underbrace{\theta^\mu(e_\beta)}_{\delta^\mu_\beta}] - e_\beta[\underbrace{\theta^\mu(e_\alpha)}_{\delta^\mu_\alpha}] - \theta^\mu([e_\alpha, e_\beta])$$

$$= 0 - 0 - c^\sigma_{\alpha\beta}\, \theta^\mu(e_\sigma) = - c^\mu_{\alpha\beta} .)$$

Also use the notation,

$$e_\alpha f = f_{,\alpha} \qquad \text{for } f \in \mathcal{F}(M).$$

Note: Nakahara avoids this, he always writes things like $e_\alpha[\Gamma^\mu_{\sigma\tau}]$ for what I will write as $\Gamma^\mu_{\sigma\tau,\alpha}$.

Now begin with (M, ∇), but don't assume any g, nor that torsion $T=0$. ~~Comp~~ First we define connection coefficients,

$$\nabla_\mu \equiv \nabla_{e_\mu}$$

$$\nabla_\mu e_\nu = \Gamma^\alpha_{\mu\nu}\, e_\alpha$$

Equivalently, $$\nabla_\mu \theta^\nu = -\Gamma^\nu_{\mu\alpha}\, \theta^\alpha .$$

Now defn of torsion,

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]. = -T(Y,X) = \text{a vector field.}$$

Components:

$$\begin{aligned} T(e_\mu, e_\nu) &= T^\alpha_{\mu\nu}\, e_\alpha \\ &= \nabla_\mu e_\nu - \nabla_\nu e_\mu - [e_\mu, e_\nu] \\ &= (\Gamma^\alpha_{\mu\nu} - \Gamma^\alpha_{\nu\mu} - C^\alpha_{\mu\nu})\, e_\alpha \end{aligned}$$

hence $$\boxed{T^\alpha_{\mu\nu} = \Gamma^\alpha_{\mu\nu} - \Gamma^\alpha_{\nu\mu} - C^\alpha_{\mu\nu}}$$

Similarly for the curvature tensor,

$$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$$

$$R(e_\alpha, e_\beta)\, e_\nu = R^\mu{}_{\nu\alpha\beta}\, e_\mu$$

Let $e'_{\alpha} = \Lambda_{\alpha}{}^{\beta}\, e_{\beta}$, defines $\Lambda_{\alpha}{}^{\beta}$. Then demand $g(e'_{\alpha}, e'_{\beta}) = \eta_{\alpha\beta}$, and you find

$$\Lambda^{\alpha}{}_{\mu}\, \eta_{\alpha\beta}\, \Lambda^{\beta}{}_{\nu} = \eta_{\mu\nu}$$

where indices are raised + lowered with η. Thus $\Lambda^{\alpha}{}_{\mu}(x)$ is an <u>x-dependent Lorentz transformation</u>. <u>These are gauge transformations in GR</u>. How other things transform:

$$\theta'^{\mu} = \Lambda^{\mu}{}_{\alpha}\, \theta^{\alpha}.$$

Any tensor transforms pointwise-linearly in $\Lambda(x)$, for example, the Riemann-Cartan 2-form,

$$R'^{\mu}{}_{\nu} = \Lambda^{\mu}{}_{\alpha}\, \Lambda_{\nu}{}^{\beta}\, R^{\alpha}{}_{\beta}.$$

But the Cartan-Connection 1-form has a less simple transformation law (since Γ is not a tensor):

$$\omega'^{\sigma}{}_{\nu} = \Lambda^{\sigma}{}_{\gamma}\, \Lambda_{\nu}{}^{\beta}\, \omega^{\gamma}{}_{\beta} - \Lambda^{\sigma}{}_{\gamma,\alpha}\, (\Lambda^{-1})^{\gamma}{}_{\nu}\, \theta^{\alpha}.$$

The extra term on the right is characteristic of the transformation laws for gauge potentials.

$$\boxed{\cancel{\omega^{\gamma}{}_{\beta} = \Lambda_{\sigma}{}^{\gamma}\, \omega'^{\sigma}{}_{\nu}\, \Lambda^{\nu}{}_{\beta} + \Lambda_{\sigma}{}^{\gamma}\, \Lambda^{\sigma}{}_{\beta,\alpha}\, \theta^{\alpha}}}$$

We found previously,

$$R^{\mu}_{\cdot\,\nu\alpha\beta} = \Gamma^{\mu}_{\beta\nu,\alpha} - \Gamma^{\mu}_{\alpha\nu,\beta} + \Gamma^{\sigma}_{\beta\nu}\Gamma^{\mu}_{\alpha\sigma} - \Gamma^{\sigma}_{\alpha\nu}\Gamma^{\mu}_{\beta\sigma} - C^{\sigma}_{\alpha\beta}\Gamma^{\mu}_{\sigma\nu}.$$

Now follow Cartan and make the following definitions:

$$\omega^{\mu}_{\cdot\,\nu} = \Gamma^{\mu}_{\alpha\nu}\,\theta^{\alpha}$$ (Lie-algebra valued 1-form)

$$T^{\mu} = \tfrac{1}{2}\,T^{\mu}_{\alpha\beta}\,\theta^{\alpha}\wedge\theta^{\beta}$$ (vector-valued 2-form)

$$R^{\mu}_{\cdot\,\nu} = \tfrac{1}{2}\,R^{\mu}_{\cdot\,\nu\alpha\beta}\,\theta^{\alpha}\wedge\theta^{\beta}$$ (Lie-algebra valued 2-form).

Now take

$$d\theta^{\mu} = -\tfrac{1}{2}\,C^{\mu}_{\alpha\beta}\,\theta^{\alpha}\wedge\theta^{\beta}$$

use components of T, elim. $C^{\mu}_{\alpha\beta}$ in favor of T, Γ,

$$= +\tfrac{1}{2}\left(T^{\mu}_{\alpha\beta} - \Gamma^{\mu}_{\alpha\beta} + \Gamma^{\mu}_{\beta\alpha}\right)\theta^{\alpha}\wedge\theta^{\beta}$$

equal

$$= T^{\mu} - \omega^{\mu}_{\cdot\,\beta}\wedge\theta^{\beta},$$

or

$$\boxed{d\theta^{\mu} + \omega^{\mu}_{\cdot\,\beta}\wedge\theta^{\beta} = T^{\mu}}$$ 1st Cartan structure eqn.

LHS is a kind of covariant derivative of a 1-form.

Next, take defn. $\omega^{\mu}_{\cdot\,\nu} = \Gamma^{\mu}_{\alpha\nu}\,\theta^{\alpha}$, apply d:

$$d\omega^{\mu}_{\cdot\,\nu} = d\left(\Gamma^{\mu}_{\alpha\nu}\,\theta^{\alpha}\right) = \Gamma^{\mu}_{\alpha\nu,\beta}\,\theta^{\beta}\wedge\theta^{\alpha} + \Gamma^{\mu}_{\alpha\nu}\,d\theta^{\alpha}$$

$$= \tfrac{1}{2}\left(\Gamma^{\mu}_{\alpha\nu,\beta} - \Gamma^{\mu}_{\beta\nu,\alpha}\right)\theta^{\beta}\wedge\theta^{\alpha} - \tfrac{1}{2}\,\Gamma^{\mu}_{\alpha\nu}\,C^{\alpha}_{\sigma\tau}\,\theta^{\sigma}\wedge\theta^{\tau}$$

$$= \frac{1}{2}\left(\Gamma^{\mu}_{\beta\nu,\alpha} - \Gamma^{\mu}_{\alpha\nu,\beta} - \Gamma^{\mu}_{\sigma\nu}\, C^{\sigma}_{\alpha\beta} \right) \theta^{\alpha}\wedge\theta^{\beta}$$

$$= \frac{1}{2}\left(R^{\mu}_{\cdot\nu\alpha\beta} - \Gamma^{\sigma}_{\beta\nu}\Gamma^{\mu}_{\alpha\sigma} + \Gamma^{\sigma}_{\alpha\nu}\Gamma^{\mu}_{\beta\sigma} \right) \theta^{\alpha}\wedge\theta^{\beta}$$

$$= R^{\mu}_{\cdot\nu} - \frac{1}{2}\,\omega^{\mu}_{\cdot\sigma}\wedge\omega^{\sigma}_{\cdot\nu} + \frac{1}{2}\,\omega^{\sigma}_{\cdot\nu}\wedge\omega^{\mu}_{\cdot\sigma},$$

equal

or,

$$\boxed{d\omega^{\mu}_{\cdot\nu} + \omega^{\mu}_{\cdot\sigma}\wedge\omega^{\sigma}_{\cdot\nu} = R^{\mu}_{\cdot\nu}}$$ 2nd Cartan structure eqn.

Again, take

$$T^{\mu} = d\theta^{\mu} + \omega^{\mu}_{\cdot\alpha}\wedge\theta^{\alpha}, \qquad \text{apply } d,$$

$$dT^{\mu} = 0 + \left(R^{\mu}_{\cdot\alpha} - \cancel{\omega^{\mu}_{\cdot\sigma}\wedge\omega^{\sigma}_{\cdot\alpha}}\right)\wedge\theta^{\alpha}$$

$$- \omega^{\mu}_{\cdot\alpha}\wedge d\theta^{\alpha} \rightarrow T^{\alpha} - \cancel{\omega^{\alpha}_{\beta}\wedge\theta^{\beta}}$$

$$\boxed{dT^{\mu} + \omega^{\mu}_{\cdot\alpha}\wedge T^{\alpha} = R^{\mu}_{\cdot\alpha}\wedge\theta^{\alpha}}$$ 1st Bianchi identity, generalized to case $T\neq 0$.

Finally, take ~~the~~ 2nd Cartan structure, apply d:

$$dR^{\mu}_{\cdot\nu} = d\omega^{\mu}_{\cdot\sigma}\wedge\omega^{\sigma}_{\cdot\nu} - \omega^{\mu}_{\cdot\sigma}\wedge d\omega^{\sigma}_{\cdot\nu}$$

$$= \left(R^{\mu}_{\cdot\sigma} - \cancel{\omega^{\mu}_{\cdot\alpha}\wedge\omega^{\alpha}_{\cdot\sigma}}\right)\wedge\omega^{\sigma}_{\cdot\nu}$$

$$- \omega^{\mu}_{\cdot\sigma}\wedge\left(R^{\sigma}_{\cdot\nu} - \cancel{\omega^{\sigma}_{\cdot\alpha}\wedge\omega^{\alpha}_{\cdot\nu}}\right)$$

one might say that the covariant exterior derivative of the curvature 2-form is 0, that this form is closed in this sense.

$$\boxed{dR^{\mu}_{\cdot\nu} + \omega^{\mu}_{\cdot\sigma} \wedge R^{\sigma}_{\cdot\nu} - R^{\mu}_{\cdot\sigma} \wedge \omega^{\sigma}_{\cdot\nu} = 0}$$

2nd Bianchi, generalized.

When $T=0$, these eqns should reduce to the previous versions of the Bianchi identities. For the 1st Bianchi ident., this gives

$$0 = R^{\mu}_{\cdot\alpha} \wedge \theta^{\alpha} = \frac{1}{2} R^{\mu}_{\cdot\nu\alpha\beta}\, \theta^{\nu} \wedge \theta^{\alpha} \wedge \theta^{\beta}$$

$$\Rightarrow R^{\mu}_{\cdot[\nu\alpha\beta]} = 0. \qquad \text{checks.}$$

For the 2nd Bianchi ident., notice that it doesn't involve T at all. But if you want to show equivalence to $R^{\mu}_{\cdot\nu[\alpha\beta;\gamma]} = 0$, you must use $T=0$.

Now consider the case that we have a metric g and a metric connection $\nabla g = 0$.

Then it is convenient to assume the basis $\{e_\alpha\}$ is <u>orthonormal</u>, i.e.,

$$g_{\alpha\beta} = g(e_\alpha, e_\beta) = \eta_{\alpha\beta} \quad (\text{pseudo-Riem. case, or } \delta_{\alpha\beta}, \text{ Riem. case}).$$
$$= \text{const. metric of special relativity.}$$

We know that if the curvature tensor $\neq 0$, then there is no <u>coordinate basis</u> such that $g_{\alpha\beta} = \eta_{\alpha\beta}$. But there are always <u>non-coordinate</u> bases that make this true. This is a special kind of vielbein.

There are some special properties of Γ, R in orthonormal vielbeins. First, $\nabla g = 0$ implies

$$0 = g_{\mu\nu,\alpha} - \Gamma^{\beta}_{\alpha\mu}\, g_{\beta\nu} - \Gamma^{\beta}_{\alpha\nu}\, g_{\mu\beta}.$$

Define $\Gamma_{\alpha\mu\nu} = g_{\alpha\beta}\,\Gamma^{\beta}{}_{\mu\nu}$. Note, this $\Gamma_{\alpha\mu\nu}$ is the 1-form index.

Also, in an orthonormal vielbein, $g_{\mu\nu} = \eta_{\mu\nu}$ so $g_{\mu\nu,\alpha} = 0$. Thus,

$$\Gamma_{\mu\alpha\nu} + \Gamma_{\nu\alpha\mu} = 0,$$

and $\Gamma_{\mu\nu}$ is antisymmetric in $\mu\nu$. (Recall in coord. basis w. LC connection, $\Gamma_{\alpha\mu\nu} = \Gamma_{\alpha\nu\mu}$.) This property depends only on $\nabla g = 0$ (the parallel transport proceed by orthogonal (or Lorentz) transformations), it does not require the LC connection.

In terms of Cartan's forms, this condition is

$$\omega_{\mu\nu} = -\,\omega_{\nu\mu} \qquad (\omega_{\mu\nu} = \eta_{\mu\alpha}\,\omega^{\alpha}{}_{\nu}).$$

Similarly, we have

$$R_{\mu\nu} = -R_{\nu\mu} \qquad (R_{\mu\nu} = \eta_{\mu\alpha}\,R^{\alpha}{}_{\nu},\ \text{Riemann-Cartan tensor})$$

for the same reason.

Also note, if in addition $T=0$ (L.C. connection) then
$$\Gamma^{\alpha}_{\mu\nu} = +\tfrac{1}{2}\left(C^{\alpha}{}_{\mu\nu} + C_{\mu}{}^{\alpha}{}_{\nu} + C_{\nu}{}^{\alpha}{}_{\mu}\right)$$

Now we consider a change of basis for an orthonormal vielbein. To be specific, we'll assume the pseudo-Riemannian (1+3) case, with $g_{\mu\nu} = \eta_{\mu\nu}$. A change of basis maps one orthonormal vielbein to another.

We are assuming that $\quad g(e_\alpha, e_\beta) = \eta_{\alpha\beta}.$

Let $\quad e'_\mu = \Lambda^{\alpha}{}_{\mu}\, e_\alpha$, and demand that $g(e'_\mu, e'_\nu) = \eta_{\mu\nu}$, so the new vielbein is also orthonormal.

in (3+1)-dim. space-time.

Now we deal with the variational formulation of GR. We work in coordinates x^μ. We start with the vacuum (matter-free) case, for which the field eqns are

$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = 0.$$

We seek a Lagrangian density $\mathcal{L}_G$ such that these eqns follow from

$$\delta \int d^4x \sqrt{-g}\ \mathcal{L}_G = 0.$$

Here $d^4x = dx^0 \wedge \ldots \wedge dx^3$, $g = \det g_{\mu\nu} < 0$, so $-g = |g|$. The product $d^4x \sqrt{-g}$ is the <u>invariant</u> <u>volume</u> <u>element</u>, as will be explained later in the course. $\mathcal{L}_G$ must be a scalar in order that the integral be independent of coordinates. The simplest scalar that can be constructed out of $g_{\mu\nu}$ and its derivatives (apart from trivial things like $g^\mu{}_\mu = 4$) is the curvature scalar R. So we guess that $\mathcal{L}_G \propto R$, and we look at the variation,

$$\delta \int d^4x \sqrt{-g}\, R = 0.$$

The variation is carried out by $g_{\mu\nu} \mapsto g_{\mu\nu} + \delta g_{\mu\nu}$. First we compute the variation in $g^{\mu\nu}$ induced by $\delta g_{\mu\nu}$. Use

$$g^{\mu\alpha} g_{\alpha\beta} = \delta^\mu_\alpha \quad \Rightarrow$$

$$\delta g^{\mu\alpha} g_{\alpha\beta} + g^{\mu\alpha} \delta g_{\alpha\beta} = 0$$

$$\Rightarrow \quad \delta g^{\mu\nu} = -g^{\mu\alpha} g^{\nu\beta} \delta g_{\alpha\beta}$$

Next we compute $\delta\sqrt{-g}$. Let M be a matrix that depends on a parameter λ. Then we have the useful identity,

$$\frac{d}{d\lambda}(\det M) = (\det M)\, \mathrm{tr}\left(M^{-1}\frac{dM}{d\lambda}\right).$$

Identify M with $g_{\mu\nu}$, $\det M = g$, this implies

$$\delta g = g\,(g^{\mu\nu}\delta g_{\mu\nu}),$$

or

$$\delta\sqrt{-g} = \tfrac{1}{2}\sqrt{-g}\,(g^{\mu\nu}\delta g_{\mu\nu}).$$

Finally, we need δR. Start ~~by~~ with $\delta\Gamma^{\mu}_{\alpha\beta}$, the change in the L.C. Γ when $g_{\mu\nu}$ goes to $g_{\mu\nu}+\delta g_{\mu\nu}$. Being the ~~change~~ difference between 2 connections, this is a tensor, which we will write as $(\delta\Gamma)^{\mu}{}_{\alpha\beta}$ to be careful about the positions of the indices. Of course $\Gamma^{\mu}_{\alpha\beta}$ itself is not a tensor.

Now we compute $\delta R^{\mu}{}_{\nu\alpha\beta}$ in terms of $\delta\Gamma$. The expression for R has the structure

$$R = \partial\Gamma - \partial\Gamma + \Gamma\Gamma - \Gamma\Gamma,$$

omitting all indices. Therefore

$$\delta R = \partial(\delta\Gamma) - \partial(\delta\Gamma) + (\delta\Gamma)\Gamma + \Gamma(\delta\Gamma) - (\delta\Gamma)\Gamma - \Gamma(\delta\Gamma).$$

We evaluate $\delta R^{\mu}{}_{\nu\alpha\beta}$ at an arbitrary point of the manifold that we call O, $\delta R^{\mu}{}_{\nu\alpha\beta}(O)$. We use Riemann normal coordinates based at O, so $\Gamma^{\mu}_{\nu\alpha}(O)=0$. Thus

$$\delta R^{\mu}{}_{\nu\alpha\beta}(O) = (\delta\Gamma)^{\mu}{}_{\beta\nu,\alpha}(O) - (\delta\Gamma)^{\mu}{}_{\alpha\nu,\beta}(O),$$

since all 4 terms in Γ-$\delta\Gamma$ vanish. Since $\delta\Gamma$ is a tensor, ~~all~~ both terms above are ordinary derivatives of tensors, evaluated at O.

But in R.N.C., such ord. derivs are equal to covariant derivs, (evaluated at 0). So we can replace the comma with a semicolon. Then we have a relation between two tensors,

$$\delta R^{\mu}_{\cdot\,\nu\alpha\beta}(0) = (\delta\Gamma)^{\mu}_{\cdot\,\beta\nu;\alpha}(0) - (\delta\Gamma)^{\mu}_{\cdot\,\alpha\nu;\beta}(0).$$

But since O was arbitrary, this is true at all points,

$$\delta R^{\mu}_{\cdot\,\nu\alpha\beta} = (\delta\Gamma)^{\mu}_{\cdot\,\beta\nu;\alpha} - (\delta\Gamma)^{\mu}_{\cdot\,\alpha\nu;\beta}$$

And since it is a tensor eqn, it is valid in all coordinates (not only RNC).

Now by contracting, we get the variation of the Ricci tensor,

$$\delta R_{\nu\beta} = (\delta\Gamma)^{\alpha}_{\cdot\,\beta\nu;\alpha} - (\delta\Gamma)^{\alpha}_{\cdot\,\alpha\nu;\beta}$$

or juggling indices

$$\delta R_{\mu\nu} = (\delta\Gamma)^{\alpha}_{\cdot\,\nu\mu;\alpha} - (\delta\Gamma)^{\alpha}_{\cdot\,\alpha\mu;\nu}.$$

Finally, as for the curvature scalar, we have $R = g^{\mu\nu}R_{\mu\nu} = R_{\mu}^{\cdot\,\mu}$,

$$\delta R = \delta g^{\mu\nu} R_{\mu\nu} + g^{\mu\nu}\,\delta R_{\mu\nu}$$

$$= -g^{\mu\alpha}g^{\nu\beta}\,\delta g_{\alpha\beta}R_{\mu\nu} + g^{\mu\nu}\left((\delta\Gamma)^{\alpha}_{\cdot\,\nu\mu;\alpha} - (\delta\Gamma)^{\alpha}_{\cdot\,\alpha\mu;\nu}\right)$$

$$\delta R = -R^{\mu\nu}\,\delta g_{\mu\nu} + (\delta\Gamma)^{\alpha\mu}_{\cdot\,\cdot\,\mu;\alpha} - (\delta\Gamma)^{\alpha}_{\cdot\,\alpha}{}^{\mu}_{\cdot\,;\mu}.$$

Thus,

$$\delta\int d^4x\,\sqrt{-g}\,R = \int d^4x\left[\delta\sqrt{-g}\,R + \sqrt{-g}\,\delta R\right]$$

$$= \int d^4x\,\sqrt{-g}\left[\tfrac{1}{2}g^{\mu\nu}R\,\delta g_{\mu\nu} - R^{\mu\nu}\delta g_{\mu\nu} + (\delta\Gamma)^{\alpha\mu}_{\cdot\,\cdot\,\mu;\alpha} - (\delta\Gamma)^{\alpha}_{\cdot\,\alpha}{}^{\mu}_{\cdot\,;\mu}\right]$$

$$= 4 \text{ terms.}$$

The last two terms vanish on integration. For example, let

$$X^{\alpha} = \delta\Gamma^{\alpha\mu}{}_{\cdot\,\mu} ,$$

so the expression

$$X^{\alpha}{}_{;\alpha}$$

(the covariant divergence of a vector) appears in the integral. This can also be written,

$$X^{\alpha}{}_{;\alpha} = \frac{1}{\sqrt{-g}} \left(\sqrt{-g}\, X^{\alpha} \right)_{,\alpha}$$

by an identity we will prove shortly. Thus

$$\int d^4x \sqrt{-g}\; X^{\alpha}{}_{;\alpha} = \int d^4x \left(\sqrt{-g}\, X^{\alpha} \right)_{,\alpha} = 0$$

by integration by parts (X vanishes at ∞). (or maybe M is compact). Similarly for the 4th term. Thus,

$$\delta \int \sqrt{-g}\, d^4x\; R = \int d^4x \sqrt{-g} \left[+\tfrac{1}{2} g^{\mu\nu} R - R^{\mu\nu} \right] \delta g_{\mu\nu} = 0$$

for all $\delta g_{\mu\nu}$ $\Rightarrow$

$$+\tfrac{1}{2} g^{\mu\nu} R - R^{\mu\nu} = -G^{\mu\nu} = 0$$

the vacuum Einstein equations.

Conventionally we take

$$\mathcal{L}_G = \frac{R}{16\pi G} ,$$

G = Newton's constant of gravitation, henceforth set to 1.

If a matter Lagrangian $\mathcal{L}_M$ is added to $\mathcal{L}_G$ and the overall variational principle is