Now, about the Cartan formalism for connection, curvature and torsion. This is discussed in the book in connection with a metric, but the main results don't depend on a metric so I'll do them here.

To summarize, there are 3 approaches to connection, torsion and curvature:

1. Intuitive and/or coordinate point of view. Useful for getting an intuitive idea of what these objects mean geometrically.
2. Approach based on ∇ operator, in general a map
$$\nabla : \mathfrak{X}(M) \times \begin{pmatrix} \text{tensor field} \\ \text{any type} \end{pmatrix} \to \begin{pmatrix} \text{same type} \\ \text{tensor field} \end{pmatrix}$$
with certain properties that allow its definition to be given for any type of tensor.
3. Cartan approach using differential forms.

We take up #3. now.

We work with an arbitrary (possibly noncoordinate) frame $\{e_\mu\}$ and dual basis of forms $\{\theta^\mu\}$. We use the notation,
$$e_\mu(f) = f_{,\mu}$$
so that
$$df = f_{,\mu}\, \theta^\mu.$$
Also, we have
$$\nabla_\alpha e_\beta = \Gamma^\mu_{\alpha\beta}\, e_\mu \qquad (\text{definition of } \Gamma^\mu_{\alpha\beta} \text{ given } \nabla)$$

and $$\nabla_\alpha \theta^\mu = - \Gamma^\mu_{\alpha\beta}\,\theta^\beta ,$$

where $$\nabla_\alpha \equiv \nabla_{e_\alpha} .$$

Beware, notation like $\nabla_\alpha e_\beta$ is used in the GR literature for something somewhat different.

The Cartan formalism is often used in a context in which $\{e_\mu\}$ is an orthonormal frame, but an orthonormal frame is not defined without a metric, so for now $\{e_\mu\}$ is an arbitrary frame.

~~The basis forms θ^μ are a collection of 1-forms indexed by μ. We can think of them as a "vector" of 1-forms. Now following Cartan, we define a "vector" of 2-forms,~~

$$\cancel{T^\mu = d\theta^\mu + \omega}$$ Not so fast

The connection coefficients $\Gamma^\mu_{\alpha\nu}$ define an element of $\mathfrak{gl}(n,\mathbb{R})$, specified by a displacent vector (intuitively, a small displacement) X^α by

$$(\Gamma_X)^\mu_{\cdot\,\nu} = X^\alpha \Gamma^\mu_{\alpha\nu}$$

Then the parallel transport of Y at x to Y' at $x+\epsilon X$ is given by

$$Y'^\mu = \left(\delta^\mu_\nu \underset{\text{(conventional - sign)}}{-} \epsilon(\Gamma_X)^\mu_{\cdot\,\nu}\right) Y^\nu .$$

So we get a Lie algebra $\mathfrak{gl}(n, \mathbb{R})$-valued 1-form if we write

$$\omega^{\mu}{}_{\nu} = \Gamma_{\alpha}{}^{\mu}{}_{\nu}\, \theta^{\alpha}$$

or we can just think of it as a matrix-valued 1-form, equivalent to the definition of a connection.

so that

$$\omega^{\mu}{}_{\nu}(X) = (\Gamma_X)^{\mu}{}_{\nu}.$$

If you like, $\omega^{\mu}{}_{\nu}$ is a matrix of 1-forms that is equivalent to $\Gamma_{\alpha}{}^{\mu}{}_{\nu}$ or ∇.

Also, the basis forms θ^{μ} can be thought of as a "vector" of 1-forms, indexed by μ. In Cartan's formalism, we consider various "scalars", "vectors", "tensors" etc. of differential r-forms, for various r.

Here is a vector of 2-forms,

$$\boxed{T^{\mu} = d\theta^{\mu} + \omega^{\mu}{}_{\nu} \wedge \theta^{\nu}.} \qquad (1)$$

It is called T^{μ} because it is essentially the torsion. To see this, convert T^{μ} to its components:

use $d\theta^{\mu} = -\frac{1}{2} C^{\mu}_{\alpha\nu}\, \theta^{\alpha} \wedge \theta^{\nu}$

and $\omega^{\mu}{}_{\nu} = \Gamma_{\alpha}{}^{\mu}{}_{\nu}\, \theta^{\alpha}$.

so $$T^{\mu} = -\frac{1}{2} C^{\mu}_{\alpha\beta}\, \theta^{\alpha} \wedge \theta^{\beta} + \Gamma_{\alpha}{}^{\mu}{}_{\nu}\, \theta^{\alpha} \wedge \theta^{\nu}$$

$$= \frac{1}{2}\left(\Gamma_{\alpha}{}^{\mu}{}_{\nu} - \Gamma_{\nu}{}^{\mu}{}_{\alpha} - C^{\mu}_{\alpha\nu}\right) \theta^{\alpha} \wedge \theta^{\nu}$$

Thus.

$$T^{\mu}_{\alpha\nu} = \Gamma^{\mu}_{\alpha\nu} - \Gamma^{\mu}_{\nu\alpha} - C^{\mu}_{\alpha\nu}.$$

This agrees with the components of T defined by (= torsion)

$$T : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$$

$$: (X, Y) \longmapsto \nabla_X Y - \nabla_Y X - [X, Y]$$

which implies

$$T(e_\alpha, e_\nu) = \left(\Gamma^{\mu}_{\alpha\nu} - \Gamma^{\mu}_{\nu\alpha} - C^{\mu}_{\alpha\nu} \right) e_\mu .$$

So, T^μ defined by (1) above is equivalent to the component or ∇-based definition of torsion. We can roughly think of T^μ as giving a "vector-valued 2-form" since for each μ we have a 2-form giving the μ-th component of $T(X,Y)$. Actually, the real vector-valued 2-form is

$$T = e_\mu \otimes T^\mu .$$

Similarly, Cartan defines a 2-rank tensor $R^{\mu}{}_{\cdot\nu}$ of 2-forms by

$$\boxed{R^{\mu}{}_{\cdot\nu} = d\omega^{\mu}{}_{\cdot\nu} + \omega^{\mu}{}_{\cdot\sigma} \wedge \omega^{\sigma}{}_{\cdot\nu}} \qquad (2)$$

As suggested by the notation, this is a version of the curvature tensor (but much easier to remember than the components of the curvature tensor).

To prove this, use ${\omega^\mu}_{\nu} = \Gamma^\mu_{\alpha\nu}\,\theta^\alpha$, so

$$d{\omega^\mu}_{\nu} = d\left(\Gamma^\mu_{\alpha\nu}\,\theta^\alpha\right)$$

$$= d\Gamma^\mu_{\alpha\nu} \wedge \theta^\alpha + \Gamma^\mu_{\alpha\nu}\, d\theta^\alpha$$

$$= \Gamma^\mu_{\alpha\nu,\beta}\,\theta^\beta \wedge \theta^\alpha + \Gamma^\mu_{\sigma\nu}\left(-\tfrac{1}{2} C^\sigma_{\alpha\beta}\,\theta^\alpha \wedge \theta^\beta\right)$$

$$= \tfrac{1}{2}\left(\Gamma^\mu_{\alpha\nu,\beta} - \Gamma^\mu_{\beta\nu,\alpha} - \Gamma^\mu_{\sigma\nu}\, C^\sigma_{\alpha\beta}\right)\theta^\alpha \wedge \theta^\beta.$$

Similarly,

$${\omega^\mu}_{\sigma} \wedge {\omega^\sigma}_{\nu} = \left(\Gamma^\mu_{\alpha\sigma}\,\theta^\alpha\right) \wedge \left(\Gamma^\sigma_{\beta\nu}\,\theta^\beta\right)$$

$$= \Gamma^\mu_{\alpha\sigma}\Gamma^\sigma_{\beta\nu}\,\theta^\alpha \wedge \theta^\beta = \tfrac{1}{2}\left(\Gamma^\mu_{\alpha\sigma}\Gamma^\sigma_{\beta\nu} - \Gamma^\mu_{\beta\sigma}\Gamma^\sigma_{\alpha\nu}\right)\theta^\alpha \wedge \theta^\beta.$$

Putting all this together and using

derived earlier from the ∇-based definition of R

$${R^\mu}_{\nu\alpha\beta} = \left(\Gamma^\mu_{\alpha\nu,\beta} + \Gamma^\mu_{\alpha\sigma}\Gamma^\sigma_{\beta\nu} - (\alpha \leftrightarrow \beta)\right) - \Gamma^\mu_{\sigma\nu}\, C^\sigma_{\alpha\beta},$$

we get

$$\boxed{{R^\mu}_{\nu} = \tfrac{1}{2}{R^\mu}_{\nu\alpha\beta}\,\theta^\alpha \wedge \theta^\beta}$$

So we see that ${R^\mu}_{\nu}$ is the curvature 2-form. It can be interpreted as a $\mathfrak{gl}(n,\mathbb{R})$-valued 2-form, since ${R^\mu}_{\nu}(X,Y)$ gives the correction to the identity when a vector is parallel-transported around a small parallelogram spanned by X, Y.

We informally refer to an object like T^μ as a "vector-valued 2-form," because at a point $x \in M$ ~~the spe~~ and $X, Y \in T_x M$ it specifies a vector $\in T_x M$ given by

$$e_\mu T^\mu(X,Y) \equiv T(X,Y).$$

which, as shown, agrees with the ∇-based definition of T, $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$. To be precise, T^μ by itself is a collection of $\mathbb{R}$-valued 2-forms, indexed by μ.

Moreover it is important to note that the product $e_\mu \otimes T^\mu$ ~~is not a tensor~~ is not ~~obviously~~ obviously a tensor, until we show that it is independent of the basis. To do this, we can perform a change of basis, $\{e_\mu\}, \{\theta^\mu\} \to \{e'_\mu\}, \{\theta'^\mu\}$, where

$$e'_\mu = e_\alpha M^\alpha{}_{\cdot\mu} \qquad (3)$$

$$\theta'^\mu = (M^{-1})^\mu{}_{\cdot\alpha}\, \theta^\alpha$$

where $M^\alpha{}_{\cdot\mu}$ is a matrix function of x with $\det M^\alpha{}_{\cdot\mu} \neq 0$ (thus, $M \in GL(n, \mathbb{R})$). If we do this to the Cartan definition of T^μ,

$$T^\mu = d\theta^\mu + \omega^\mu{}_{\cdot\nu} \wedge \theta^\nu, \qquad (2)$$

we find that

$$e_\mu \otimes T^\mu = e'_\mu \otimes T'^\mu,$$

so T^μ is a tensor. ~~[illegible]~~ If we do this, we find that each term in (2) is not a tensor, but the sum is. We will not carry out this calculation, since we have shown that the components of the Cartan definition of T^μ and the ∇-definition of T are the same, and we have shown (in book, lecture and notes) that the ∇-definition of T is a tensor.

Similarly, the Cartan definition $R^{\mu}{}_{\cdot\nu} = d\omega^{\mu}{}_{\cdot\nu} + \omega^{\mu}{}_{\cdot\sigma} \wedge \omega^{\sigma}{}_{\cdot\nu}$ is a tensor-valued 2-form (i.e., the components of a tensor valued 2-form) because we have shown that the Cartan defn of $R^{\mu}{}_{\cdot\nu}$ and the ∇-definition of R have the same components, while we have shown elsewhere (book, lecture, notes) that the latter is a tensor.

Similarly, the frame forms θ^μ constitute the (components of) a vector valued 1-form, i.e., a linear map at each $x \in T_xM$ of $T_xM \to T_xM$. In fact the map is just the identity, since

$$(e_\mu \otimes \theta^\mu)(X) = e_\mu\, \theta^\mu(X) = e_\mu X^\mu = X ,$$

and

$$e_\mu \otimes \theta^\mu = e'_\mu \otimes \theta'^\mu .$$

This expression is like the resolutions of the identity $\sum_n |n\rangle\langle n|$ that occur in quantum mechanics.

The connection 1-forms $\omega^{\mu}_{\cdot\nu}$, however, are not a tensor, precisely because $\Gamma^{\mu}_{\alpha\nu}$ is not (the components of) a tensor. In fact, under the transformation (3), we find

$$\omega'^{\mu}_{\cdot\nu} = (M^{-1})^{\mu}_{\cdot\alpha} M^{\beta}_{\cdot\nu} \omega^{\beta}_{\cdot\alpha} + (M^{-1})^{\mu}_{\cdot\alpha} dM^{\alpha}_{\cdot\nu} .$$

The presence of the second term means that $\omega^{\mu}_{\cdot\nu}$ is not a tensor. This is basically a version of the transformation law for $\Gamma^{\mu}_{\alpha\nu}$, which is not a tensor, either.

When we take a (genuine) vector-valued form, such as θ^{μ}, and compute the exterior derivatives of the components, to get $d\theta^{\mu}$, the result is not a tensor, that is, $e_{\mu}\otimes d\theta^{\mu}$ is not independent of the choice of frame. But if we add the term $\omega^{\mu}_{\cdot\nu}\wedge\theta^{\nu}$, as in the definition of T^{μ}, the non-tensorial parts of the 2 terms cancel, and the result is a tensor.

This is similar to what happens when computing the covariant derivative of a vector Y^{μ} in components. It is

$$Y^{\mu}_{\ ;\nu} = Y^{\mu}_{\ ,\nu} + \Gamma^{\mu}_{\alpha\nu} Y^{\nu}$$

Neither term on the RHS is a tensor, but the sum is.

This leads us to define an "exterior covariant derivative" D, a generalization of the "ordinary" exterior derivative d. Like d, D maps r-forms into (r+1)-forms, but d acts only on scalar valued forms while D acts on any tensor-valued form. An ordinary differential form is considered a scalar-valued form, and for these $D = d$. But for a vector-valued form, $D = d$ + a correction term involving ω, seen in the definition of T:

$$T^\mu = d\theta^\mu + \omega^\mu{}_\nu \wedge \theta^\nu = D\theta^\mu.$$

Since θ^μ is a vector-valued 1-form, T^μ is a vector-valued 2-form.

For another example, an ordinary vector Y^μ can be regarded as a vector-valued 0-form. Then DY^μ is a vector valued 1-form, given by

$$DY^\mu = dY^\mu + \omega^\mu{}_\nu Y^\nu.$$

Thus

$$\begin{aligned}(DY^\mu)(X) &= X^\nu Y^\mu{}_{,\nu} + X^\alpha \Gamma^\mu_{\alpha\nu} Y^\nu \\ &= X^\nu Y^\mu{}_{;\nu} = (\nabla_X Y)^\mu\end{aligned}$$

More generally, D applied to any ordinary tensor, interpreted as a tensor valued 0-form, is the same as ∇.

In general, when D acts on a tensor-valued r-form, it produces the ordinary exterior derivative plus one correction term for each index of the tensor. This correction term can be seen in the definition $T^\mu = D\theta^\mu = d\theta^\mu + \omega^\mu{}_\nu \wedge \theta^\nu$.

Similarly, we can compute DT^μ:

$$DT^{\mu} = DD\theta^{\mu} = dT^{\mu} + \omega^{\mu}{}_{\cdot\nu} \wedge T^{\nu}$$

$$= d\left(d\theta^{\mu} + \omega^{\mu}{}_{\cdot\nu} \wedge \theta^{\nu}\right) + \omega^{\mu}{}_{\cdot\nu} \wedge T^{\nu}$$

$$= 0 + d\omega^{\mu}{}_{\cdot\nu} \wedge \theta^{\nu} - \omega^{\mu}{}_{\cdot\nu} \wedge d\theta^{\nu} + \omega^{\mu}{}_{\cdot\nu} \wedge \left(d\theta^{\nu} + \omega^{\nu}{}_{\cdot\sigma} \wedge \theta^{\sigma}\right)$$

cancel.

$$= \left(d\omega^{\mu}{}_{\cdot\nu} + \omega^{\mu}{}_{\cdot\sigma} \wedge \omega^{\sigma}{}_{\cdot\nu}\right) \wedge \theta^{\nu},$$

or

$$\boxed{DT^{\mu} = DD\theta^{\mu} = R^{\mu}{}_{\cdot\nu} \wedge \theta^{\nu}}$$

We see that DD is not zero, in general, unlike DD.

Similarly, D applied to a 2-nd rank tensor-valued form produces 2 correction terms. For example,

$$DR^{\mu}{}_{\cdot\nu} = dR^{\mu}{}_{\cdot\nu} + \omega^{\mu}{}_{\cdot\sigma} \wedge R^{\sigma}{}_{\cdot\nu} - \omega^{\sigma}{}_{\nu} \wedge R^{\mu}{}_{\sigma}$$

But this is

$$d\left(d\omega^{\mu}{}_{\cdot\nu} + \omega^{\mu}{}_{\cdot\sigma} \wedge \omega^{\sigma}{}_{\cdot\nu}\right)$$

$$+\,\omega^{\mu}{}_{\cdot\sigma} \wedge \left(d\omega^{\sigma}{}_{\cdot\nu} + \omega^{\sigma}{}_{\cdot\tau} \wedge \omega^{\tau}{}_{\cdot\nu}\right)$$

$$-\,\omega^{\sigma}{}_{\cdot\nu} \wedge \left(d\omega^{\mu}{}_{\cdot\sigma} + \omega^{\mu}{}_{\cdot\tau} \wedge \omega^{\tau}{}_{\cdot\sigma}\right)$$

$$= 0 + d\omega^{\mu}{}_{\cdot\sigma} \wedge \omega^{\sigma}{}_{\cdot\nu} - \omega^{\mu}{}_{\cdot\sigma} \wedge d\omega^{\sigma}{}_{\cdot\nu}$$

cancel. cancel

$$+\,\omega^{\mu}{}_{\cdot\sigma} \wedge d\omega^{\sigma}{}_{\cdot\nu} + \omega^{\mu}{}_{\cdot\sigma} \wedge \omega^{\sigma}{}_{\cdot\tau} \wedge \omega^{\tau}{}_{\cdot\nu}$$ cancel.

$$-\,\omega^{\sigma}{}_{\cdot\nu} \wedge d\omega^{\mu}{}_{\cdot\sigma} - \omega^{\sigma}{}_{\cdot\nu} \wedge \omega^{\mu}{}_{\cdot\tau} \wedge \omega^{\tau}{}_{\cdot\sigma} \qquad = 0.$$

Thus we derive

$$\boxed{DR^{\mu}{}_{\cdot\nu} = 0}$$

<u>Summary</u>. An <u>tensor-valued r-form</u> has components that are real-valued r-forms. These include scalar, vector, and higher rank tensors. A scalar-valued r-form is an ordinary r-form (it has no indices). A tensor-valued 0-form is an ordinary tensor.

The exterior covariant derivative D is a generalization of the ordinary exterior derivative d. D maps a tensor valued r-form into a tensor-valued (r+1)-form, the same type of tensor. The components of D of a tensor valued r-form are d of the components, plus one correction term involving ω with a + sign for every upper index, and one correction involving ω with a − sign for every lower index. The rules are best explained by examples. θ^{μ} is a vector-valued 1-form (really the components of a vector-valued one-form). Then

$$(D\theta)^{\mu} = d\theta^{\mu} + \omega^{\mu}{}_{\cdot\nu} \wedge \theta^{\nu} \equiv T^{\mu} \quad \text{(torsion)}$$

$$(DT)^{\mu} = dT^{\mu} + \omega^{\mu}{}_{\cdot\nu} \wedge T^{\nu}$$

$$(DR)^{\mu}{}_{\cdot\nu} = dR^{\mu}{}_{\cdot\nu} + \omega^{\mu}{}_{\cdot\sigma} \wedge R^{\sigma}{}_{\cdot\nu} - \omega^{\sigma}{}_{\cdot\nu} \wedge R^{\mu}{}_{\sigma}$$

These rules resemble the rules for computing the components of the ~~[illegible]~~ covariant derivative of tensors. For example, in the case of a vector,

$$(\nabla_\alpha X)^\mu = X^\mu_{,\alpha} + \Gamma^\mu_{\alpha\beta} X^\beta$$

$$(\nabla_\alpha g)_{\mu\nu} = g_{\mu\nu,\alpha} - \Gamma^\sigma_{\alpha\mu} g_{\sigma\nu} - \Gamma^\sigma_{\alpha\nu} g_{\mu\sigma} \quad = g_{\mu\nu;\alpha}$$

In fact, the rules are the same, if we interpret an ordinary tensor as a tensor-valued 0-form, and define D as the same as ∇ on these. For example,

$$(\nabla g)_{\mu\nu} = (D g)_{\mu\nu} = d g_{\mu\nu} - \omega^{\sigma}_{\cdot\mu} g_{\sigma\nu} - \omega^{\sigma}_{\cdot\nu} g_{\mu\sigma}$$

which is a tensor valued 1-form. If we let this act on $X \in \mathfrak{X}(M)$, we get $(Dg)_{\mu\nu}(X) = X^\alpha g_{\mu\nu;\alpha}$.

The definition of R,

$$R^\mu_{\cdot\nu} = d\omega^\mu_{\cdot\nu} + \omega^\mu_{\cdot\sigma} \wedge \omega^\sigma_{\cdot\nu}$$

looks superficially like "$R = D\omega$" but it's not, because $\omega^\mu_{\cdot\nu}$ is not a tensor and if it were $D\omega$ would have 2 correction terms, not one.

The two equations,

$$DT^\mu = R^\mu_{\cdot\nu} \wedge \theta^\nu$$

and $$DR^\mu_{\cdot\nu} = 0$$

are essentially the <u>first and second</u> <u>Bianchi identities</u>, although one usually finds these identities in the literature expressed in components under the assumption $T^\mu = 0$ (because $T^\mu = 0$ is a standard assumption in GR).

For the first Bianchi identity in coordinates, assume $T^\mu = 0$ and write

$$R^{\mu}_{\cdot\nu} = \tfrac{1}{2} R^{\mu}_{\cdot\nu\alpha\beta}\, \theta^\alpha \wedge \theta^\beta .$$

Then

$$\tfrac{1}{2} R^{\mu}_{\cdot\nu\alpha\beta}\, \theta^\alpha \wedge \theta^\beta \wedge \theta^\nu = 0,$$

or, since $\theta^\alpha \wedge \theta^\beta \wedge \theta^\nu$ is completely antisymmetric in the 3 indices,

$$\boxed{R^{\mu}_{\cdot[\nu\alpha\beta]} = 0}$$

1st Bianchi; when $T = 0$

For the second Bianchi identity, it can be shown that

$$0 = DR^{\mu}_{\cdot\nu} = \tfrac{1}{2} R^{\mu}_{\cdot\nu\alpha\beta;\sigma}\, \theta^\sigma \wedge \theta^\alpha \wedge \theta^\beta + R^{\mu}_{\cdot\nu\alpha\beta}\, T^\alpha \wedge \theta^\beta .$$

So, if $T^\mu = 0$, this implies

$$\boxed{R^{\mu}_{\cdot\nu[\alpha\beta;\sigma]} = 0.}$$

2nd Bianchi; when $T = 0$.