So we can solve for $(\nabla_X \omega)_\nu$, get

$$\boxed{(\nabla_X \omega)_\nu = X^\mu (\omega_{\nu,\mu} - \Gamma^\alpha_{\mu\nu}\, \omega_\alpha)}$$

c.f. earlier result for vectors

$$\boxed{(\nabla_X Y)^\nu = X^\mu (Y^\nu_{,\mu} + \Gamma^\nu_{\mu\alpha}\, Y^\alpha)}$$

Similarly can work out rules for covariant derivatives (in components) for an arbitrary tensor. Basically you get an ordinary derivative with one correction term with Γ and a + sign for every contravariant index, and one correc. term with Γ and a − sign for every covariant index. For example, you find for the metric tensor,

$$(\nabla_X g)_{\mu\nu} = X^\alpha (g_{\mu\nu,\alpha} - \Gamma^\beta_{\alpha\mu}\, g_{\beta\nu} - \Gamma^\beta_{\alpha\nu}\, g_{\mu\beta})$$

Note, also have

$$\boxed{\nabla_\mu \, dx^\nu = - \Gamma^\nu_{\mu\alpha}\, dx^\alpha}$$

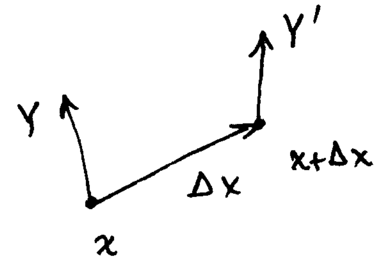
Now we turn to the transformation properties of the connection coefficients $\Gamma^\mu_{\alpha\beta}$. Basic fact is that $\Gamma^\mu_{\alpha\beta}$ is not a tensor. A tensor is a mapping of vectors and covectors onto scalars, that is point-wise linear (linear at each point). We can think of Γ as such a mapping,

$$\Gamma : \mathfrak{X}^*(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{F}(M) : (\alpha, X, Y) \mapsto \alpha(\nabla_X Y),$$

$$\Gamma^\mu_{\alpha\beta} = dx^\mu (\nabla_\alpha e_\beta) \qquad e_\beta = \frac{\partial}{\partial x^\beta}.$$

But it is not point-wise linear in the Y operand (it depends on the derivatives of Y as well as the value of Y at a point). Here are various ways to see this.

① Consider the parallel transport of Y from x to $x+\Delta x$,



We have

$$Y'^{\mu} = \underbrace{(\delta^{\mu}_{\nu} - \Delta x^{\alpha}\, \Gamma^{\mu}_{\alpha\nu})}\, Y^{\nu}$$

→ a near-identity element of $GL(n,\mathbb{R})$, so we can think of

$$\Gamma^{\mu}{}_{\nu} = dx^{\alpha}\, \Gamma^{\mu}_{\alpha\nu} \quad \text{as a } \underline{\mathfrak{gl}(n,\mathbb{R})\text{-valued 1-form}}.$$

But notice that the components of this 1-form depend on the basis chosen in two different tangent spaces (at x and $x+\Delta x$). You can change one without changing the other. Hence $\Gamma^{\mu}_{\alpha\nu}$ does not transform as a tensor.

To emphasize this, consider the following fact: The difference between two connections, say, $\Gamma - \bar{\Gamma}$, is a tensor. That's because $\Gamma - \bar{\Gamma}$ can be thought of as specifying the parallel transport from x to $x+\Delta x$, using Γ, then back again, using $\bar{\Gamma}$. The vector is transported from one tangent space back to the same tangent space. (Say, $Y \to Y' \to Y''$). Then

$$Y''^{\mu} = (\delta^{\mu}_{\nu} + dx^{\alpha}\, \Gamma^{\mu}_{\alpha\nu} - dx^{\alpha}\, \bar{\Gamma}^{\mu}_{\alpha\nu})\, Y^{\nu}.$$

Thus, only one basis (in T_xM) need be chosen to specify the near-identity element of $GL(n,\mathbb{R})$ mapping Y to Y''.

(2) Just do a brute-force transformation of the connection coefficients. Let

$$e_\alpha = \frac{\partial}{\partial x^\alpha} \qquad e'_\mu = \frac{\partial}{\partial x'^\mu}, \qquad \begin{array}{l} \nabla_\alpha = \nabla_{e_\alpha} \\ \nabla'_\mu = \nabla_{e'_\mu} \end{array}$$

$$\Gamma^{\alpha}_{\beta\gamma} = (dx^\alpha, \nabla_\beta e_\gamma)$$

$$= \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial^2 x'^\mu}{\partial x^\beta \partial x^\gamma} + \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x'^\sigma}{\partial x^\gamma}\frac{\partial x'^\nu}{\partial x^\beta}\Gamma'^{\mu}_{\nu\sigma}$$

The 2nd term looks like a tensor transformation law, but the first term spoils it (and involves 2nd derivatives of the coordinate transformation). But if you subtract the transformation laws for two Γ's, say, $\Gamma - \bar{\Gamma}$, then the first term cancels.

Transformation laws like this are familiar for the gauge potential A^a_μ of gauge-field theories (Yang-Mills, QCD).

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besides subtracting $\Gamma - \bar{\Gamma}$

Notice that another way to cancel the first term is to antisymmetrize in (β,γ). This leads to a tensor called the torsion:

$$T^{\alpha}{}_{\beta\gamma} = \Gamma^{\alpha}{}_{\beta\gamma} - \Gamma^{\alpha}{}_{\gamma\beta}.$$

This is the component definition of the torsion. The coordinate-free definition is

$$T : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M) : (X, Y) \mapsto \nabla_X Y - \nabla_Y X - [X, Y]$$

This is obviously antisymmetric. To show that it is a tensor, we must show that it is pointwise linear in each operand (but due to the antisymmetry, we need only check one). Let $f \in \mathcal{F}(M)$. Then

$$T(fX, Y) = \nabla_{fX} Y - \nabla_Y (fX) - [fX, Y]$$

$$= f \nabla_X Y - (Yf)X - f \nabla_Y X - f XY + (Yf)X + f YX$$

cancel

$$= f\left[\nabla_X Y - \nabla_Y X - [X, Y] \right]$$

$$= f\, T(X, Y).$$

So it's a tensor. Now let $\{e_\mu\}$ be any basis of vector fields (coordinate or non-coordinate). Then we define the components of T by

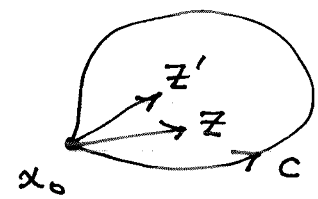
$$T(e_\alpha, e_\beta) = T^{\mu}{}_{\alpha\beta}\, e_\mu$$

$$= \nabla_\alpha e_\beta - \nabla_\beta e_\alpha - [e_\alpha, e_\beta]$$

$$= \left(\Gamma^{\mu}_{\alpha\beta} - \Gamma^{\mu}_{\beta\alpha} - C^{\mu}_{\alpha\beta} \right) e_\mu ,$$

where $C^{\mu}_{\alpha\beta}$ are the "structure constants", i.e. $[e_\alpha, e_\beta] = C^{\mu}_{\alpha\beta}\, e_\mu$. (Really $C^{\mu}_{\alpha\beta}$ depend on x, in general). This agrees with earlier coordinate-based definition of T, where we used a coordinate basis so that $C^{\mu}_{\alpha\beta} = 0$.

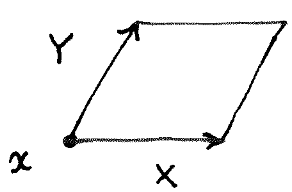
Now we take up <u>curvature</u> and <u>holonomy</u>. Consider the parallel transport of a vector $Z \in T_{x_0}M$ around a loop c based at x_0:



This produces a linear map: $T_{x_0}M \to T_{x_0}M : Z \mapsto Z'$ (in the picture). The map is invertible, since each infinitesimal step of the parallel transport is an invertible map between neighboring tangent spaces. Thus, the map is an element of $GL(n, \mathbb{R})$. Call the map P_c (it is parameterized by the loop c.) P_c is called the <u>holonomy</u> of the loop.

Notice in general $P_c \in GL(n,\mathbb{R})$, but if a metric exists and a metric connection is employed, then P_c preserves scalar products, i.e., $P_c : T_xM \to T_xM$ is an orthogonal transformation (a member of $SO(n)$ for an orientable, Riemannian manifold, or $SO(n,m)$ for an oriented, pseudo-Riem. manifold). In general, the set of all possible holonomies of all possible loops based at x_0 is a subgroup of $GL(n,\mathbb{R})$, called the holonomy group at x_0, denoted $H(x_0)$. Like the fundamental group, elements of the holonomy group depend on the loop, but they are not invariant under continuous deformation. If points x_0 and x_1 can be connected by a curve as above, then $H(x_0)$ and $H(x_1)$ are conjugate groups, $H(x_0) = \tau\, H(x_1)\, \tau^{-1}$. As abstract groups they are the same. Then one can speak of the holonomy group of the manifold. For example, the holonomy group of the 2-sphere (under the Levi-Civita connection and the obvious metric) is $SO(2)$.

If the loop is infinitesimal then we get an infinitesimal element of the holonomy group, i.e., an element of the Lie algebra. E.g., consider an infinitesimal parallelogram defined by vectors X and Y:



Then the Lie algebra element you get upon parallel transporting around the small loop depends on the area element (it is linear and antisymmetric in X, Y), i.e., it is a Lie algebra-valued 2-form:

$$R : \underbrace{T_xM \times T_xM}_{\text{antisymm.}} \longrightarrow \text{Lie algebra of } H(x)\text{, e.g., } \mathfrak{gl}(n,\mathbb{R}),\ so(n).$$

Conventions for attaching indices to R. Let the Lie algebra element be represented by an $n \times n$ matrix, in some basis in $T_x M$. Then write,

conventional minus sign

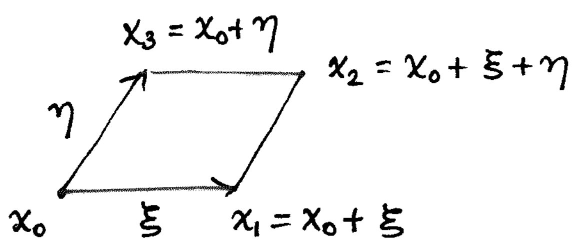
$$Z'^{\mu} = \left[\delta^{\mu}_{\nu} - R(X,Y)^{\mu}_{\cdot\nu} \right] Z^{\mu}$$

for the parallel transport of Z around the X–Y parallelogram (along X first, then Y). The correction term is linear and antisymmetric in X, Y, hence

$$R(X,Y)^{\mu}_{\cdot\nu} = \underbrace{R^{\mu}{}_{\nu\alpha\beta}}_{\text{curvature tensor.}} X^{\alpha} Y^{\beta}$$

where $R^{\mu}_{\cdot\nu\alpha\beta} = -R^{\mu}_{\cdot\nu\beta\alpha}$.

How to calculate $R^{\mu}_{\cdot\nu\alpha\beta}$ in a coordinate basis $e_{\mu} = \frac{\partial}{\partial x^{\mu}}$. Change notation slightly, write ξ, η instead of X, Y (ξ, η are infinitesimals). These define an infinitesimal parallelogram in the given coordinates,



The sides of the parallelogram are straight lines in the given coordinates. Thus, on transporting a vector Z along the first leg $x_0 \to x_1$, we create a curve parametrized by t, $x^{\mu}(t) = x^{\mu}_0 + t\,\xi^{\mu}$, $0 \le t \le 1$.

Notation: Let $(\xi \cdot \Gamma)$ be the $n \times n$ matrix with components,

$$\underbrace{(\xi \cdot \Gamma)^{\mu}_{\cdot\nu}}_{(\Gamma_{\xi})^{\mu}_{\cdot\nu}} = \xi^{\alpha}\, \Gamma^{\mu}_{\alpha\nu}.$$

or Γ_{ξ}

Eqn. of parallel transport is

$$\frac{dZ^\mu}{dt} = -\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{dt} Z^\beta . \qquad \text{But} \quad x^\alpha(t) = x_0^\alpha + t\,\xi^\alpha$$

$$\frac{dx^\alpha}{dt} = \xi^\alpha$$

so $$\frac{dZ^\mu}{dt} = -\underbrace{(\Gamma_\xi)^\mu_{\cdot\beta}}_{\text{eval. at } x(t).} Z^\beta ,$$

or $$\frac{dZ}{dt} = -\Gamma_\xi(x_0 + t\xi)\, Z \qquad \text{for short.} \quad = Z'$$

then $$\frac{d^2Z}{dt^2} = -\xi\cdot\nabla\Gamma_\xi\, Z - \Gamma_\xi \frac{dZ}{dt} = Z''.$$

$$= \left(-\xi\cdot\nabla\Gamma_\xi + \Gamma_\xi^2\right) Z \qquad \text{where} \quad \xi\cdot\nabla\Gamma_\xi = \xi^\mu(\Gamma_\xi)_{,\mu}.$$

So, $$Z_0' = -\Gamma_\xi Z_0$$

$$Z_0'' = \left(-\xi\cdot\nabla\Gamma_\xi + \Gamma_\xi^2\right) Z_0 .$$

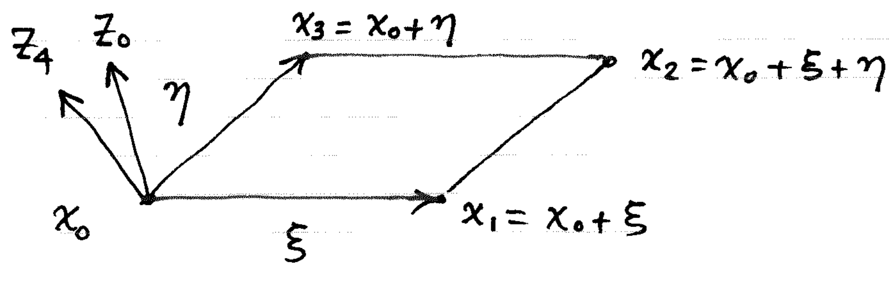
So, $$Z_1 = \left[\mathrm{Id} - \Gamma_\xi + \tfrac{1}{2}\left(-\xi\cdot\nabla\Gamma_\xi + \Gamma_\xi^2\right)\right] Z_0 \qquad \text{Taylor series at } t=1.$$

everything in [ ] eval at x_0. Z_1 = value of Z, parallel transported from x_0 to x_1. To transport $x_1 \to x_2$, replace $Z_0 \to Z_1 \to Z_2$, $\xi \to \eta$, $x_0 \to x_1 = x_0 + \xi$. Thus,

$$Z_2 = \Big[\mathrm{Id} \underbrace{- \Gamma_\eta(x_0+\xi)}_{\to\ -\Gamma_\eta(x_0) - \xi\cdot\nabla\Gamma_\eta} + \tfrac{1}{2}\left(-\eta\cdot\nabla\Gamma_\eta + \Gamma_\eta^2\right)\Big] Z_1$$

Summary...

Computation of curvature tensor. Given manifold M with connection ∇, but not nec. anything else (such as g). Parallel transport a vector Z around the 4 sides of a parallelogram spanned by two infinitesimal vectors ξ, η, with corners x_0, x_1, x_2, x_3, giving $Z_0 \to Z_1 \to Z_2 \to Z_3 \to Z_4$, where $Z_0, Z_4 \in T_{x_0}M$:



The transport $Z_0 \to Z_4$ must be linear and near-identity (since the parallelogram is small). Write it

$$Z_4^{\mu} = [\mathrm{Id} - R(\xi,\eta)]^{\mu}{}_{\nu}\, Z_0^{\nu}$$

where $R(\xi,\eta)^{\mu}{}_{\nu}$ is linear in ξ, η, so

$$R(\xi,\eta)^{\mu}{}_{\nu} = R^{\mu}{}_{\nu\alpha\beta}\, \xi^{\alpha}\eta^{\beta}$$

defines the components $R^{\mu}{}_{\nu\alpha\beta}$ of the curvature tensor. It is anti-symmetric in ξ, η, since if $\xi = \eta$, then you are parallel transporting along a line and back again, which cancels (gives $R(\xi,\xi) = 0$). Thus we expect

$$R^{\mu}{}_{\nu\alpha\beta} = -R^{\mu}{}_{\nu\beta\alpha}.$$

Equivalently, R is a <u>Lie-algebra valued 2-form.</u> Of course we must verify these expected properties of R (such as the fact that it is a tensor).

On the leg $x_0 \to x_1$, the //-transport equation can be solved in a Taylor series in the small displacement ξ, which we expand through 2nd order:

not covariant deriv., just shorthand.

$$Z_1 = \left[Id - \Gamma_\xi(x_0) + \tfrac{1}{2}\left(- \xi\cdot\nabla\Gamma_\xi + \Gamma_\xi^2 \right) \right] Z_0$$

shorthand for $Z_1^\mu = [\ \ ]^\mu_{\cdot\nu} Z_0^\nu$

where $(\Gamma_\xi)^\mu_{\cdot\nu} = \xi^\alpha \Gamma^\mu_{\alpha\nu}$

$$(\xi\cdot\nabla\Gamma_\xi)^\mu_{\cdot\nu} = \xi^\beta \xi^\alpha \Gamma^\mu_{\alpha\nu,\beta}.$$

Similarly,

$$Z_3 = \left[\mathrm{Id} + \Gamma_\xi (x_0 + \xi + \eta) + \tfrac{1}{2}\left(-\xi\cdot\nabla\Gamma_\xi + \Gamma_\xi^2 \right) \right] Z_2$$

$$Z_4 = \left[\mathrm{Id} + \Gamma_\eta (x_0 + \eta) + \tfrac{1}{2}\left(-\eta\cdot\nabla\Gamma_\eta + \Gamma_\eta^2 \right) \right] Z_3 .$$

Now multiply matrices,

$$Z_4 = \underbrace{[\ ][\ ][\ ][\ ]}\, Z_0$$

$$= \mathrm{Id} - \left[\Gamma_\xi \Gamma_\eta - \Gamma_\eta \Gamma_\xi + \xi\cdot\nabla\Gamma_\eta - \eta\cdot\nabla\Gamma_\xi \right]$$

$$= \mathrm{Id} - R(\xi, \eta).$$

From this can read off components of R,

$$\boxed{R^{\mu}{}_{\cdot\,\nu\alpha\beta} = \Gamma^{\mu}_{\alpha\sigma}\Gamma^{\sigma}_{\beta\nu} - \Gamma^{\mu}_{\beta\sigma}\Gamma^{\sigma}_{\alpha\nu} + \Gamma^{\mu}_{\beta\nu,\alpha} - \Gamma^{\mu}_{\alpha\nu,\beta}}$$

Expression of curvature tensor in terms of connection, in a coordinate basis $e_\mu = \partial/\partial x^\mu$. (No metric required.)

Γ is not a tensor, but R should be a tensor, based on its definition (its a mapping of $T_x M$ onto itself, given ξ and η). A direct proof that R transforms as a tensor is tedious, however.

A coordinate-free approach is better. We define

$$R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M),$$

notation $R(X, Y, Z) = \underbrace{R(X,Y)}\, Z$

will turn out to be same $R(X,Y)$ defined above where

$$R(X,Y) : \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M),$$

$$\boxed{R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}}$$

coordinate-free defn of R.

pointwise

First show that this is a tensor. Must be linear in X, Y. Because of antisymmetry, suffices to check only X. Let $f \in \mathcal{F}(M)$

$$R(fX,Y) = \underbrace{\nabla_{fX}\nabla_Y}\ \underbrace{- \ \nabla_Y \nabla_{fX}}\ \underbrace{- \ \nabla_{[fX,Y]}}$$

$$f\,\nabla_X \nabla_Y$$

$$-\nabla_Y f \nabla_X = -\cancel{(Yf)}\nabla_X - f\,\nabla_Y \nabla_X$$

$$= -\nabla_{f[X,Y] - (Yf)X} = -f\,\nabla_{[X,Y]} + \cancel{(Yf)}\nabla_X$$

$$= f\,R(X,Y).$$

should also be linear in Z. Check it.

$$R(X,Y)fZ = \nabla_X \nabla_Y fZ - \nabla_Y \nabla_X fZ - \nabla_{[X,Y]} fZ$$

$$= T_1 + T_2 + T_3 \qquad \text{(3 terms).}$$

$$T_1 = \nabla_X (Yf) Z + \nabla_X f \nabla_Y Z$$

$$= (XYf) Z + (Yf) \nabla_X Z + (Xf) \nabla_Y Z + f \nabla_X \nabla_Y Z$$

$T_2 = T_1$ with $X \leftrightarrow Y$, minus sign.

$$= -(YXf) Z - (Xf) \nabla_Y Z - (Yf) \nabla_X Z - f \nabla_Y \nabla_X Z$$

$$T_3 = -([X,Y]f) Z - f \nabla_{[X,Y]} Z$$

add em up, get $f R(X,Y) Z$.

So R is a tensor.

Now compute the components of R (starting from coordinate-free definition) and compare to earlier coordinate-based calculation. For variety do this in a non-coordinate basis e_μ $\left(\neq \frac{\partial}{\partial x^\mu}\right)$.

Define:

① $f_{,\mu} = (e_\mu f)$ when $f \in \mathcal{F}(M)$, generalizes notation $f_{,\mu} = e_\mu f = \frac{\partial f}{\partial x^\mu}$ for a coordinate basis.

② $\nabla_\mu = \nabla_{e_\mu}$ ← Be careful! ∇_μ notation is widely used in GR literature for something slightly different from this.

③ $\nabla_\alpha e_\beta = \Gamma^\mu_{\alpha\beta} e_\mu$

④ $R(e_\alpha, e_\beta) e_\nu = R^\mu{}_{\nu\alpha\beta}\, e_\mu$

$$R(e_\alpha, e_\beta)\, e_\nu = \nabla_\alpha \nabla_\beta e_\nu - \nabla_\beta \nabla_\alpha e_\nu - \nabla_{[e_\alpha, e_\beta]} e_\nu$$

$$= \nabla_\alpha (\Gamma^\sigma_{\beta\nu} e_\sigma) = \Gamma^\sigma_{\beta\nu,\alpha} e_\sigma + \Gamma^\sigma_{\beta\nu} \Gamma^\mu_{\alpha\sigma} e_\mu$$

$=$ − same w. $(\alpha \leftrightarrow \beta)$.

$$= -\nabla_{c^\sigma_{\alpha\beta} e_\sigma} e_\nu = -c^\sigma_{\alpha\beta} \nabla_\sigma e_\nu = -c^\sigma_{\alpha\beta} \Gamma^\mu_{\sigma\nu} e_\mu$$

$c^\sigma_{\alpha\beta}$ = structure consts of basis.

gives

$$\boxed{R^\mu{}_{\nu\alpha\beta} = \Gamma^\mu_{\beta\nu,\alpha} + \Gamma^\sigma_{\beta\nu} \Gamma^\mu_{\alpha\sigma} - \Gamma^\mu_{\alpha\nu,\beta} - \Gamma^\sigma_{\alpha\nu}\Gamma^\mu_{\beta\sigma} - c^\sigma_{\alpha\beta}\Gamma^\mu_{\sigma\nu}}$$

agrees with earlier calculation in coord. basis, for which $c^\sigma_{\alpha\beta} = 0$.

In the case of the Levi-Civita connection on a (pseudo)-Riemannian manifold, the curvature tensor is also called the <u>Riemann tensor</u>. I'm not sure if that is appropriate in other cases.

The curvature tensor has various <u>symmetries</u>, depending on the assumptions. In the most general case (manifold M + connection ∇, nothing else) we have the symmetry,

$$R(X,Y) = -R(Y,X) \quad \text{or} \quad R^\mu{}_{\nu\alpha\beta} = -R^\mu{}_{\nu\beta\alpha}.$$

This just says that R is a <u>2-form</u> (indices α, β). That's all in this case.

If in addition we assume torsion $T=0$, then there are two further symmetries, called the 1st and 2nd Bianchi identities. The first Bianchi identity is an algebraic condition on R. In components it is

$$R^{\mu}{}_{[\nu\alpha\beta]} = 0$$ where [ ] means complete antisymmetrization,

i.e. $R^{\mu}{}_{\nu\alpha\beta} + R^{\mu}{}_{\alpha\beta\nu} + R^{\mu}{}_{\beta\nu\alpha} = 0$ in this case, since R is already antisymmetric in $(\alpha\beta)$. This statement is equivalent to

$$R(X,Y)Z + \circlearrowleft = 0$$

where $\circlearrowleft$ means to cycle X,Y,Z (X,Y correspond to indices $\alpha\beta$, the 2-form indices, while Z corresponds to ν, in $R^{\mu}{}_{\nu\alpha\beta}$). We will prove this in coordinate free form. First use the definition of $R(X,Y)$ to write

$$R(X,Y)Z + \circlearrowleft = \nabla_X \boxed{\nabla_Y Z} - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \, .$$

↑ substitute

But since $T=0$, $T(Y,Z) = \nabla_Y Z - \nabla_Z Y - [Y,Z] = 0$, so

$$\text{RHS} = \underbrace{\nabla_X \nabla_Z Y}_{} + \nabla_X [Y,Z] - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z + \circlearrowleft$$

$X \to Y \to Z$

becomes $\nabla_Y \nabla_X Z$

cancel.

$$\text{RHS} = \nabla_X [Y,Z] - \nabla_{[X,Y]} Z + \circlearrowleft$$

But $T(X,[Y,Z]) = \nabla_X [Y,Z] - \nabla_{[Y,Z]} X - [X,[Y,Z]] = 0$ so

0 Jacobi

$$\text{RHS} = \nabla_{[Y,Z]} X - \underbrace{\nabla_{[X,Y]} Z}_{\hookrightarrow\ \nabla_{[Y,Z]} X} + \cancel{[X,[Y,Z]]} + \circlearrowleft$$

cancel

$= 0$ QED.

The 2nd Bianchi identity is a differential equation satisfied by $R^\mu{}_{\nu\alpha\beta}$. Since R is a 2-form (indices $\alpha\beta$), we might ask what the exterior derivative is and whether it is interesting. If R were a scalar-valued 2-form instead of a Lie-algebra-valued 2-form, we might compute (in components)

$$R^\mu{}_{\nu[\alpha\beta,\gamma]}$$

which would be the components of something like "dR". But since R is a Lie-algebra-valued form, it turns out we must replace the comma by a semicolon.

Digression on semicolon notation. If T is a type (r,s) tensor and X is a vector field, then $\nabla_X T$ is also a type (r,s) tensor. But the resulting object is point-wise linear in X, so removing the X it corresponds to a type $(r, s+1)$ tensor, call it ∇T, such that

$$\nabla_X T = i_X(\nabla T)$$

where i_X means, "contract X with one index of ∇T". For example, if $T = Y =$ a vector field $=$ a type $(1,0)$ tensor, then ∇Y is a type $(1,1)$ tensor, and we write its components by

$$(\nabla Y)^\mu{}_{\cdot\alpha} = (\nabla_\alpha Y)^\mu = \theta^\mu(\nabla_{e_\alpha} Y)$$

$$\equiv Y^\mu{}_{;\alpha}$$

Thus

$$\nabla_X Y = X^\alpha Y^\mu_{;\alpha} \quad \text{and}$$

$$Y^\mu{}_{;\alpha} = Y^\mu{}_{,\alpha} + \Gamma^\mu_{\alpha\beta} Y^\beta .$$

To return to the 2nd Bianchi identity, it is

$$R^{\mu}{}_{\nu[\alpha\beta;\gamma]} = 0.$$

In coordinate free form this is

$$(\nabla_Z R)(X,Y) + \circlearrowleft = 0$$

where X, Y correspond to $\alpha\beta$ and Z to γ. To prove this, let W be a vector field so that $R(X,Y)W$ is another vector field, apply ∇_Z to the latter and use Leibnitz:

swap, change sign

$$\nabla_Z R(X,Y)W = (\nabla_Z R)(X,Y)W + R(\nabla_Z X, Y)W + R(X, \nabla_Z Y)W$$
$$+ R(X,Y)\nabla_Z W.$$

Now cancel W to get an equation among operators acting on vector fields, and rewrite it as

$$(\nabla_Z R)(X,Y) = [\nabla_Z, R(X,Y)] - R(\nabla_Z X, Y) + R(\nabla_Z Y, X)$$

Now add cyclic pmuts and use defn of $R(X,Y)$,

0 Jacobi

$$(\nabla_Z R)(X,Y) + \circlearrowleft = \cancel{[\nabla_Z, [\nabla_X, \nabla_Y]]} - [\nabla_Z, \nabla_{[X,Y]}]$$
$$- R(\nabla_Z X, Y) + R(\nabla_Z Y, X) + \circlearrowleft$$

Now use $T(Z,X) = 0 \Rightarrow \nabla_Z X = \nabla_X Z + [Z,X]$,

$R(\nabla_X Z, Y)$

cancel. cycle set

$$\text{RHS} = -[\nabla_Z, \nabla_{[X,Y]}] - \cancel{R(\nabla_X Z, Y)} - R([Z,X], Y) + \overbrace{R(\nabla_Z Y, X)}^{\text{cycle set}}$$
$$+ \circlearrowleft$$

↑ use defn of R

cancel $[\nabla_{[X,Z]}, \nabla_Z]$

0 Jacobi

$$= -[\nabla_Z, \nabla_{[X,Y]}] - [\nabla_{[Z,X]}, \nabla_Y] + \cancel{\nabla_{[[X,Z],Y]}} + \circlearrowleft = 0 \quad \text{QED}$$