$f \circ c$. If you think of c as a map from $\mathbb{R}^r$ to M (actually it is a linear comb. of such maps, the singular cubes or simplexes) then $f \circ c$ is a map : $\mathbb{R}^r \to N$. Then we have the following fact:

$$\boxed{\int_{f\circ c} \omega = \int_c f^*\omega .}$$

This is easily proved by resorting to the definition of the integral. This definition proceeds by pulling everything back to $\mathbb{R}^r$, so it doesn't matter if we pull it back in stages or all at once, $(f\circ c)^* = c^* \circ f^*$.

Now we turn to potentials. If a r form ω can be written

$$\omega = d\psi$$

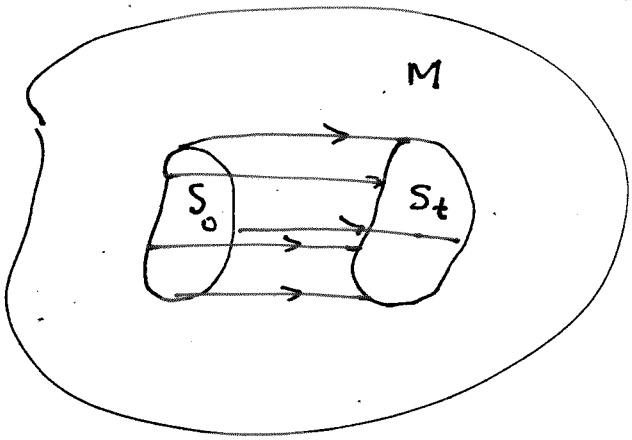
for some $(r-1)$ form ψ, then we say ψ is a potential for ω. Only exact forms have (global) potentials; this is just the meaning of "exact".

The identity $dd = 0$ means that exact $\Rightarrow$ closed. But closed $\not\Rightarrow$ exact, as we see from the monopole example. At least, this is true in a global sense on most manifolds. But it turns out that closed always $\Rightarrow$ exact locally. This is called the Poincaré lemma by Nakahara. This comes in several versions that we will explore.

First version. Let $\omega \in Z^r(M)$, $d\omega = 0$, Then for every $p \in M$ there exists a neighborhood (nbhd) such that $\exists$ an $(r-1)$-form ψ such that $\omega = d\psi$ on this neighborhood. (Continued after a long digression.)

A note on differential forms and moving chains.

If $X \in \mathfrak{X}(M)$ is a vector field and $\Phi_t = e^{tX}$ the corresponding flow, then we can use the flow to carry along a surface (or submanifold) of M:



$$S_t = \Phi_t S_0$$

S_0 is the surface at $t=0$, S_t its image at a later time. There are many applications of this idea.

In integration theory we have chains, which are linear combinations of "singular" maps:

$$c \in C_r(M),$$

$$c = \sum_i a_i \sigma_i^r, \qquad a_i \in \mathbb{R}$$

$$\sigma_i^r : I^r \to M$$

where $I = [0,1]$. The σ_i^r are r-dimensional "singular cubes", i.e. smooth maps from the r-cube I^r to M, which need not have maximal rank or be injective. The image of one σ^r is like the surface S_0 above,

in fact if you want to integrate over a submanifold of M you break it up into subsets that are the images of singular maps σ (for example, by triangulating the surface).

If $\Phi_t : M \to M$ is an advance map and

$$c = \sum_i a_i \sigma_i^r,$$

define

$$\Phi_t \circ c = \sum_i a_i \; \Phi_t \circ \sigma_i^r.$$

Each factor $\Phi_t \circ \sigma_i^r$ is a smooth map : $I^r \to M$, and so is a singular r-cube.

Now let $\omega \in \Omega^r(M)$ and consider

$$\int_{\Phi_t \circ c_0} \omega = \int_{c_0} \Phi_t^* \omega = \int_{c_t} \omega \qquad \text{where } c_t = \Phi_t \circ c_0$$

where $c_0 \in C_r(M)$ is an "initial" r-chain. Here we have used the composition rule of integrals of forms. From this we get

$$\frac{d}{dt} \int_{c_t} \omega = \int_{c_0} \frac{d}{dt} \Phi_t^* \omega .$$

Now we know that

$$\frac{d}{dt}\Big|_{t=0} \Phi_t^* = \mathcal{L}_X \qquad \text{(the Lie derivative)}$$

since this is the definition of $\mathcal{L}_X$. (This is when acting on forms.) But what is the formula when we don't have the $|_{t=0}$? The answer is easy,

$$\frac{d}{dt}\Phi_t^* = \lim_{\varepsilon \to 0}\left(\frac{\Phi_{t+\varepsilon}^* - \Phi_t^*}{\varepsilon}\right)$$

$$= \lim_{\varepsilon \to 0}\left(\frac{\Phi_t^* \Phi_\varepsilon^* - \Phi_t^*}{\varepsilon}\right)$$

$$= \lim_{\varepsilon \to 0}\left(\frac{\Phi_\varepsilon^* - 1}{\varepsilon}\right)\Phi_t^* = \lim_{\varepsilon \to 0} \Phi_t^*\left(\frac{\Phi_\varepsilon^* - 1}{\varepsilon}\right)$$

$$= \boxed{\mathcal{L}_X \Phi_t^* = \Phi_t^* \mathcal{L}_X = \frac{d}{dt}\Phi_t^*}$$

Here we use the composition rule of advance maps.

Thus

$$\frac{d}{dt}\int_{C_t} \omega = \int_{C_0} \frac{d}{dt}\Phi_t^* \omega = \int_{C_0} \Phi_t^* \mathcal{L}_X \omega$$

$$= \int_{\Phi_t \circ C_0} \mathcal{L}_X \omega = \int_{C_t} i_X d\omega + \int_{C_t} d i_X \omega.$$

This can also be written,

$$\frac{d}{dt}\int_{C_t}\omega = \int_{C_t} i_X d\omega + \int_{\partial C_t} i_X \omega$$

A useful formula. Summary: $C_t = \Phi_t \circ C_0$ $\Phi_t = e^{tX}$.

Some notes on geometrical mechanics. Let M be the configuration space of a classical system. It need not be $\mathbb{R}^n$, for example, the configuration space of a rigid body with one point fixed is $SO(3)$. If the center of mass of the rigid body is free to move, then the configuration space is $SO(3) \times \mathbb{R}^3$. Let x^i, $i = 1, \ldots, n$ be coordinates on M, where $n = \dim M$ is called the number of degrees of freedom. The x^i need not be Cartesian coordinates, they can be any coordinates. Many books on mechanics would use the symbol q^i for the coordinates but we'll call them x^i.

As explained in the notes on Noether's theorem, the Lagrangian is not a function on M, it's a function on TM, where the coordinates are (x^i, v^i). A point of TM is a tangent vector to M at a point $x \in M$ with coordinates x^i. The tangent vector itself is

$$v^i \left.\frac{\partial}{\partial x^i}\right|_x$$

Sometimes we write $\dot{x}^i$ instead of v^i, if it's more clear.

The Euler-Lagrange equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) = \frac{\partial L}{\partial x^i} .$$

In elementary mechanics these can be solved for the accelerations to give differential equations of the form,

$$\ddot{x}^i = F^i(x, v)$$

where F^i are some functions. This is not a vector field on M, (because it is second order in time),

but it can be converted into a vector field on TM, if we write $\dot{x}^i = v^i$. Then the equations of motion are

$$\left.\begin{aligned} \dot{x}^i &= v^i \\ \dot{v}^i &= F^i(x,v) \end{aligned}\right\}.$$

The vector field on TM that this corresponds to is

$$X = v^i \frac{\partial}{\partial x^i} + F^i(x,v) \frac{\partial}{\partial v^i} \qquad (A)$$

which is the operator d/dt when applied to a function $f(x,v)$, i.e., a scalar field $f: TM \to \mathbb{R}$. (For example, we might be interested in $f = \frac{1}{2}mv^2$ or $f = m(xv_y - yv_x)$ or simply $f = x$). Note that this vector (A) lies in the bundle $T(TM)$, an iterated bundle over M. Actually (A) is a vector field on TM, so it belongs to $\mathfrak{X}(TM)$.

In elementary mechanics the "momentum conjugate to x^i" is defined as

$$p_i = \frac{\partial L}{\partial \dot{x}^i}$$

by which definition it emerges as a function of $(x^i, \dot{x}^i)$. As explained earlier, we interpret this as a map $F: TM \to T^*M$, the Legendre map, that takes (x,v) to (x,p), where

$$p = p_i \, dx^i|_x.$$

The Legendre map does not change the x coordinate. Thus if it is restricted to a single fiber $T_xM \subset TM$, it becomes a map

$$F|_x : T_x M \longrightarrow T_x^* M$$

The Legendre map $F|_x$ has maximal rank ($= n = \dim M$) if

$$\det \frac{\partial p_i}{\partial \dot{x}^j} = \det \frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j} \neq 0.$$

Lagrangians that satisfy this condition everywhere on TM are said to be <u>regular</u>. For a regular Lagrangian, the rank of $F|_x$ is n and of F itself, $2n$. Regular Lagrangians are the norm in nonrelativistic mechanics, but irregular Lagrangians are the norm in relativistic mechanics and field theory.

For now let's assume the Lagrangian is regular. Then the Legendre map is a diffeomorphism, and all geometry on TM (vector fields etc) can be pushed over to T^*M.

Following the usual procedure in mechanics, we define the Hamiltonian as

$$H = p_i \dot{x}^i - L,$$

where $L = L(x, \dot{x})$ and $p_i = \partial L / \partial \dot{x}^i = p_i(x, \dot{x})$. This makes H a function on TM. But it is customary in mechanics to eliminate v^i in favor of p_i, which we can do if F is a diffeomorphism. This makes H a function of (x^i, p_i), which is the usual point of view. Geometrically, we have pushed H foward from TM to T^*M using F^{-1*}.

We can also push foward the vector field (A) from TM to T^*M (using F_*). Then the mechanics books show that this vector field is

$$\left.\begin{aligned} \dot{x}^i &= \frac{\partial H}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial H}{\partial x^i} \end{aligned}\right\} .$$

These are Hamilton's equations. They are equivalent to the vector field

$$X_H = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial x^i} - \frac{\partial H}{\partial x^i}\frac{\partial}{\partial p_i} ,$$

which is called a Hamiltonian vector field. We put an H subscript on X_H to show which Hamiltonian function this is associated with. The advance map of a Hamiltonian vector field is called a Hamiltonian flow.

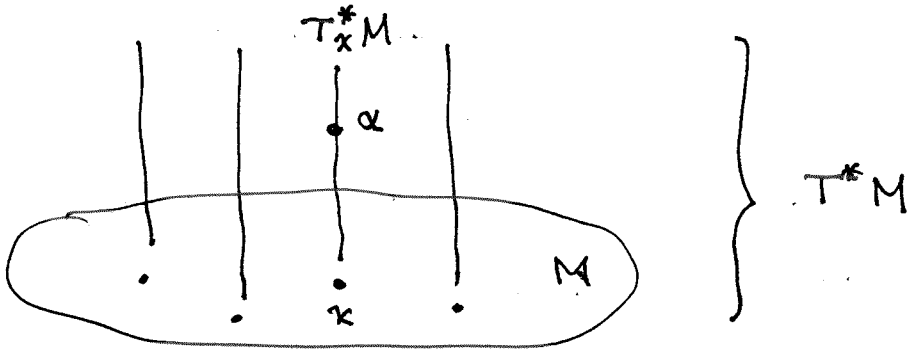
Hamilton's equations obviously give us a way to go from a scalar field $H : T^*M \to \mathbb{R}$, i.e., $H \in \mathcal{F}(T^*M) = \Omega^0(T^*M)$, to vector fields, $H \mapsto X_H \in \mathfrak{X}(T^*M)$. Notice that X_H is a vector field on T^*M, not M. It is a section of the iterated bundle $T(T^*M)$. The function we start with (here H) need not be the Hamiltonian in a physical sense; for example, if we get the vector field associated with a component of the angular momentum, it is the infinitesimal generator of rotations on T^*M. Thus we are interested quite generally in this map : $\mathcal{F}(TM) \to \mathfrak{X}(TM)$ which is represented by Hamilton's equations. What does it mean geometrically?

The answer involves the fact that every cotangent bundle carries a natural symplectic structure. Here the word "natural" means the structure arises from the geometry inherent in a cotangent bundle, without any additional structure. This construction can be used for any cotangent bundle, so for now let us forget about mechanics and let M be any manifold.

Let α be a covector attached to point $x \in M$, so that α can be written

$$\alpha = \alpha_i \, dx^i|_x ,$$

where x^i are coordinates on M. Coordinates on TM are (x^i, α_i). Sometime we'll write (x, α) instead of just α, to indicate a point of T^*M.



The projection map is $\pi : T^*M \to M : (x, \alpha) \longmapsto x$.

The geometry of the cotangent bundle allows us to define a 1-form $\theta \in \Omega^1(T^*M)$. Notice that this is not $\Omega^1(M)$, which would be a field of 1-forms over M, that is, a section of T^*M. Our θ will be a field of

1-forms over T^*M, that is, a section of $T^*(T^*M)$. In the coordinates (x^i, α_i), such a 1-form can be written,

$$\theta = a_i\, dx^i + b^i\, d\alpha_i, \qquad (C)$$

for some coefficients a_i and b^i that depend in general on (x^i, α_i). We will find these coefficients for our special form.

To define θ at a point $(x,\alpha) \in T^*M$, we need to show what it does to an arbitrary vector $X \in T_{(x,\alpha)}(T^*M)$. Such a vector can be written,

$$X = A^i \frac{\partial}{\partial x^i} + B_i \frac{\partial}{\partial \alpha_i}, \qquad (B)$$

where the partial derivs are evaluated at (x,α).

The projection is a map $\pi: T^*M \to M$, so $\pi(x,\alpha) = x$. This means that the tangent map π_* evaluated at (x,α) is a map

$$\pi_*: T_{(x,\alpha)}(T^*M) \longrightarrow T_xM.$$

π is a "vertical" projection, so π_* throws away the vertical parts of X in (B), which are the terms in $\partial/\partial\alpha_i$. (Think: $\partial/\partial\alpha_i$ means vary α_i while holding all other coordinates fixed. Holding x^i fixed means moving purely along a fiber, i.e., in the vertical direction.)

Therefore,

$$\pi_* X = \pi_* \left(A^i \left.\frac{\partial}{\partial x^i}\right|_{(x,\alpha)} + B^i \left.\frac{\partial}{\partial \alpha_i}\right|_{(x,\alpha)} \right)$$

$$= A^i \left.\frac{\partial}{\partial x^i}\right|_x \quad \in \quad T_x M. \qquad (D)$$

Now the definition of θ at (x,α) is

$$\boxed{\theta|_{(x,\alpha)}(X) = \alpha(\pi_* X),} \qquad \forall \; X \in T_{(x,\alpha)}(T^*M).$$

This is the geometrical (i.e., natural) definition of θ. According to (C) and (B), the left hand side of this eqn is

$$A^i a_i + B^i_{\cdot} b^i = \theta|_{(x,\alpha)}(X).$$

But according to (D) the right hand side is

$$\alpha(\pi_* X) = \alpha_i A^i$$

These must be equal for all X (i.e., all A^i, B_i), so $a_i = \alpha_i$ and $b^i = 0$. Thus we have,

$$\boxed{\theta = \alpha_i \, dx^i} \qquad (E)$$

If this geometrical argument is too hard to follow, it is easy to remember the result. The point is that this is a natural construction, and you must remember that $\theta \in \Omega^1(T^*M)$, not $\Omega^1(M)$. (θ superficially looks like

a 1-form on M, which would be a linear combination of $dx^i|_x$, but it is not, since α_i in (E) is not a function of x. Also, the dx^i in (E) means $dx^i|_{(x,\alpha)}$ not $dx^i|_x$.).

Closely associated with $\theta \in \Omega^1(T^*M)$ is

$$\omega = d\theta = d\alpha_i \wedge dx^i \quad \in \Omega^2(T^*M).$$

This is a closed 2-form on T^*M, since $d\omega = dd\theta = 0$.

It is also <u>nondegenerate</u>, which means the following. Let z^μ be coordinates on T^*M, for example $z^\mu = (x^i, \alpha_i)$, $i = 1, \ldots, n = \dim M$, $\mu = 1, \ldots, 2n = \dim T^*M$. Then the components of ω w.r.t. these coordinates are

$$\omega_{\mu\nu} = \omega\left(\frac{\partial}{\partial z^\mu}, \frac{\partial}{\partial z^\nu}\right).$$

To say that ω is nondegenerate means $\det \omega_{\mu\nu} \neq 0$. Thus, the component matrix is invertible if ω is nondegenerate.

nondegenerate

A 2-form ω can be used in much the same manner as a metric g with components

$$g_{\mu\nu} = g\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right)$$

where x^μ are coordinates on the manifold where g is defined. In particular, $\det g_{\mu\nu} \neq 0$ just as $\det \omega_{\mu\nu} \neq 0$ if ω is nondegenerate. Also, g at a point x can be regarded as a map from vectors to covectors. In components, this is

$$g|_x : T_xM \to T_x^*M \qquad (F)$$

$$: X^\mu \frac{\partial}{\partial x^\mu}\Big|_x \longmapsto g_{\mu\nu} X^\nu \, dx^\mu|_x$$

or, as we say, we have "lowered the index" on X^μ to give $g_{\mu\nu} X^\nu$, usually written X_μ. Here M is any manifold with coordinates x^μ and metric g. Also, since $\det g_{\mu\nu} \neq 0$, the map (F) is invertible; we write $g^{-1} : T_x^*M \to T_xM$ for the inverse map. It takes a 1-form $A_\mu \, dx^\mu|_x$ into the vector $g^{\mu\nu} A_\nu \, \partial/\partial x^\mu|_x$, that is, we have "raised the index". Here $g^{\mu\nu}$ is the inverse matrix of $g_{\mu\nu}$,

$$g_{\mu\sigma} \, g^{\sigma\nu} = \delta_\mu^\nu .$$

It is also the components of a tensor,

$$g^{\mu\nu} = (g^{-1})(dx^\mu, dx^\nu).$$

Since $\det \omega_{\mu\nu} \neq 0$, we can use ω to "raise and lower indices" on a symplectic manifold. A <u>symplectic manifold</u> is a manifold P endowed with a closed, nondegenerate 2-form. As we have seen, every cotangent bundle (i.e., $P = T^*M$) is a symplectic manifold, with $\omega = d\alpha_i \wedge dx^i$.

Let us return to mechanics, where M is the configuration

space and T^*M has coordinates (x^i, p_i) (writing p_i instead of α_i for the components of a 1-form). Let us write $P = T^*M$; P is the phase space of the system, and it is a symplectic manifold. Notice that the symplectic structure of P does not depend on the Lagrangian or Hamiltonian of the system; in that sense it is independent of mechanics. We will let $z^\mu = (x^i, p_i)$ be coordinates on P, and we'll write $z \in P$ for a point of P.

It turns out that Hamilton's equations amount to using the inverse of the symplectic form $\omega_{\mu\nu}$ to map dH (the differential of the Hamiltonian) into the Hamiltonian vector field X_H. That is, we "raise the index" on dH.

If we follow the custom with a metric, we would denote the components of ω^{-1} (the inverse of the map $\omega|_z : T_z P \to T_z^* P$) by $\omega^{\mu\nu}$ (with upper indices). I prefer to use a different symbol $J^{\mu\nu}$, that is, let's define $J^{\mu\nu}$ by

$$J^{\mu\sigma}\omega_{\sigma\nu} = \delta^\mu_\nu .$$

J may be called the Poisson tensor, since it is responsible for Poisson brackets.

Then it turns out that Hamilton's equations are equivalent to

$$\dot{z}^\mu = J^{\mu\nu} \frac{\partial H}{\partial z^\mu} .$$

To check this in components, write $\theta = p_i dx^i = \theta_\mu dz^\mu$. Then

$$\theta_\mu = \begin{pmatrix} p_i \\ 0 \end{pmatrix}$$

that is, corresponding to coordinates (x^i, p_i). Then

$$\omega_{\mu\nu} = (d\theta)_{\mu\nu} = \theta_{\nu,\mu} - \theta_{\mu,\nu} = \begin{pmatrix} 0 & -\delta_{ij} \\ +\delta_{ij} & 0 \end{pmatrix},$$

which shows that $\det \omega = 1$, so ω is symplectic (closed and nondegenerate. Then the inverse matrix is

$$J^{\mu\nu} = \begin{pmatrix} 0 & \delta_{ij} \\ -\delta_{ij} & 0 \end{pmatrix},$$

and Hamilton's equations are

$$\begin{pmatrix} \dot{x}_i \\ \dot{p}_i \end{pmatrix} = \begin{pmatrix} 0 & \delta_{ij} \\ -\delta_{ij} & 0 \end{pmatrix} \begin{pmatrix} \partial H/\partial x^i \\ \partial H/\partial p_i \end{pmatrix},$$

which is correct.

This can be written in coordinate-free form. Start with

$$X_H = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial x^i} - \frac{\partial H}{\partial x^i}\frac{\partial}{\partial p_i} \qquad (G)$$

and $\omega = dp_i \wedge dx^i$, and compute $i_{X_H}\omega$. First

$$i_{\partial/\partial x^i}(dp_j \wedge dx^j) = (i_{\partial/\partial x^i} dp_j)\, dx^j - dp_j\,(i_{\partial/\partial x^i} dx^j)$$

using the rule for the distribution of the interior product across a wedge product. But

$$i_{\partial/\partial x^i} dp_j = dp_j\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial x^i} p_j = 0$$

$$i_{\partial/\partial x^i} dx^j = \frac{\partial}{\partial x^i} x^j = \delta^j_i.$$

So,

$$i_{X_H}\omega = -\frac{\partial H}{\partial p_i} dp_i - \frac{\partial H}{\partial x^i} dx_i = -dH.$$

In summary, the relation between a Hamiltonian function H and the Hamiltonian vector field X_H is

$$\boxed{i_{X_H}\omega = -dH}$$

This is a modern version of Hamilton's equations.

The Hamiltonian vector field X_H (Eq. (G) above) is the d/dt operator when acting on functions of (x,p), i.e., <u>classical observables</u>. If $B \in \mathcal{F}(P)$, then

$$X_H B = \frac{dB}{dt} = \frac{\partial B}{\partial x^i}\frac{\partial H}{\partial p_i} - \frac{\partial B}{\partial p_i}\frac{\partial H}{\partial x_i} = \{B, H\},$$

which defines the Poisson bracket,

$$\{\ ,\ \} : \mathcal{F}(P) \times \mathcal{F}(P) \longrightarrow \mathcal{F}(P).$$

In tensor-component language,

$$(X_H)^\mu = J^{\mu\nu}\frac{\partial H}{\partial z^\nu},$$

$$X_H B = (X_H)^\mu \frac{\partial B}{\partial z^\mu} = \frac{\partial B}{\partial z^\mu} J^{\mu\nu} \frac{\partial H}{\partial z^\nu} = \{B, H\}.$$

Thus, thinking of the Poisson tensor J as a map,

$$J : \mathcal{X}^*(P) \times \mathcal{X}^*(P) \longrightarrow \mathcal{F}(P),$$

we have

$$\{B, H\} = J(dB, dH)$$

and

$$J^{\mu\nu} = J(dz^\mu, dz^\nu).$$

The Poisson bracket has the following properties:

1. Antisymmetry, $\{A, B\} = -\{B, A\}$
2. Linearity over $\mathbb{R}$ in each operand,

$$\{a_1 A_1 + a_2 A_2, B\} = a_1\{A_1, B\} + a_2\{A_2, B\}$$

$$\{A, b_1 B_1 + b_2 B_2\} = b_1\{A, B_1\} + b_2\{A, b_2\}$$

$a_1, a_2, b_1, b_2 \in \mathbb{R}$, $A, B, A_1, B_1, A_2, B_2 \in \mathcal{F}(P)$.

3. Jacobi identity,

$$\{A,\{B,C\}\} + \{B,\{C,A\}\} + \{C,\{A,B\}\} = 0.$$

4. Leibnitz,

$$\{AB,C\} = A\{B,C\} + \{A,C\}B.$$

These are the same rules as for the commutator in quantum mechanics, except in 4 the ordering of the factors does not matter.

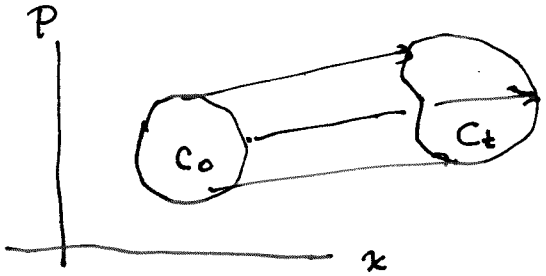
Properties (1)-(3) means that the set $\mathcal{F}(P)$, regarded as a real vector space, is a Lie algebra under the Poisson bracket. Also, as we have seen, Hamilton's equations specify a map: $\mathcal{F}(P) \to \mathcal{X}(P)$, and we know that $\mathcal{X}(P)$ is a Lie algebra under the Lie bracket. What is the relation between these two Lie algebras? The answer is given by

$$[X_A, X_B] = -X_{\{A,B\}},$$

showing that the map $\mathcal{F}(P) \to \mathcal{X}(P)$ given by Hamilton's equations is a Lie algebra anti-homomorphism. This formula is equivalent to the Jacobi identity.

Now an application to the integral invariants of Poincaré.

Let $c_0 \in C_2(P)$ (a 2-chain on phase space). You can think of this as a piece of a 2D submanifold of P. In the case $n=1$ (one degree of freedom), this is a region of the x-p plane (assuming $M=\mathbb{R}$, for a particle in 1D).



Now let each point of c_0 flow under the Hamiltonian advance map Φ_t, to produce a new chain $c_t = \Phi_t \circ c_0$. Then

$$\int_{c_t} \omega = \int_{c_0} \omega .$$

This is called an integral invariant. To prove we use our formula for moving chains,

$$\frac{d}{dt}\int_{c_t} \omega = \int_{c_t} i_x d\omega + \int_{c_t} d i_x \omega .$$

But $d\omega = 0$ so the first term vanishes. and in the second term, $i_x\omega = -dH$ (this is Hamilton's equations) so $d i_x\omega = -ddH = 0$. So $\int_{c_t} \omega$ is independent of t.

If $n=1$, then $\int_{c_0} \omega$ is the area of the region c_0. So

the theorem states that phase space area is preserved under the flow. This is Liouville's theorem for $n=1$.

The essence of this proof is

$$\begin{aligned} \mathcal{L}_X \omega &= i_X d\omega + d i_X \omega \\ &= 0 \oplus - dd H = 0. \end{aligned}$$

For $n>1$, Liouville's theorem refers to the $2n$-volume in phase space. Let us define

$$\Omega = \underbrace{\omega \wedge \omega \wedge \dots \wedge \omega}_{n\text{-times}},$$

so Ω is a $2n$-form on phase space. It's easy to show that Ω is proportional to $dx^1 \wedge \dots \wedge dx^n \wedge dp_1 \wedge \dots \wedge dp_n$, the nonzero volume form on P. Now

$$\begin{aligned} \mathcal{L}_X \Omega = \; & \mathcal{L}_X \omega \wedge \omega \wedge \dots \wedge \omega \\ & + \omega \wedge \mathcal{L}_X \omega \wedge \dots \wedge \omega \\ & + \dots \qquad = 0 \end{aligned}$$

since $\mathcal{L}_X \omega = 0$. Thus the integral of Ω over a $2n$-dim'l region that moves under a Hamiltonian flow is constant.