Then

$$f \circ g_i : I^2 \to M$$

is the i-th face, a <u>singular 2-cube</u>. This defines $\partial : C_r(M) \to C_{r-1}(M)$ (or ∂_r, more precisely)

Then we define, as in homology theory,

$$Z_r(M) = \{ c \in C_r(M) \mid \partial c = 0 \} = r\text{-th cycle group} = \ker \partial_r$$

$$B_r(M) = \{ c \in C_r(M) \mid c = \partial b, \text{ some } b \in C_{r+1} \} = r\text{-th boundary group} = \operatorname{im} \partial_{r+1}.$$

And we have $\partial^2 = 0$, as before, so $B_r(M) \subset Z_r(M)$. And we define

$$H_r(M) = \frac{Z_r(M)}{B_r(M)}. \quad r\text{-th homology group.}$$

This is same group as before.

The properties of ∂ on chains is mirrored in the properties of d acting on forms. The terminology reflects this:

$d : \Omega^r(M) \to \Omega^{r+1}(M)$ (or d_r)

$$\Omega^r(M) = \{ r\text{-forms on } M \}$$

closed forms $\to$ $Z^r(M) = \{ \omega \in \Omega^r(M) \mid d\omega = 0 \}$, r-th <u>cocycle group</u>. $= \ker d_r$

exact forms $\to$ $B^r(M) = \{ \omega \in \Omega^r(M) \mid \omega = d\beta, \text{ some } \beta \in \Omega^{r-1}(M) \}$. $= \operatorname{im} d_{r-1}$
r-th <u>coboundary group</u>

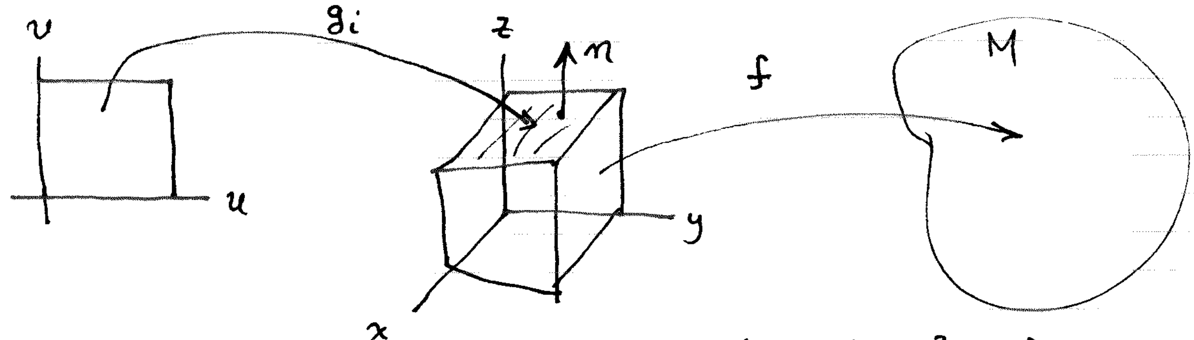
And because $d^2 = 0$, we have $B^r(M) \subset Z^r(M)$. And we define

$$H^r(M) = \frac{Z^r(M)}{B^r(M)} = \frac{\text{closed}}{\text{exact}} = \frac{\text{cocycles}}{\text{coboundaries}} = r\text{-th } \underline{\text{cohomology group}}$$

To explore this association, we need <u>Stokes' theorem</u>, which says, if $c \subset C_{r+1}(M)$, $\omega \in \Omega^r(M)$,

$$\boxed{\int_{\partial c} \omega = \int_c d\omega}$$

To prove this it suffices to consider a <u>single</u> <u>singular r-cube</u>, since chains are lin. comb's. of such things. Will do example of 3-forms.
Let $\dim M = m =$ anything. Let $\omega \in \Omega^2(M)$, so $d\omega \in \Omega^3(M)$.



$c =$ chain $(f : I^3 \to M)$.

Let $\alpha = f^*\omega$, $\alpha \in \Omega^2(\mathbb{R}^3)$. This means α has 3 nonzero components,

$$\alpha = \alpha_x \, dy \wedge dz + \alpha_y \, dz \wedge dx + \alpha_z \, dx \wedge dy,$$

$$d\alpha = d(f^*\omega) = f^*(d\omega)$$

$$= \left(\frac{\partial \alpha_x}{\partial x} + \frac{\partial \alpha_y}{\partial y} + \frac{\partial \alpha_z}{\partial z} \right) dx \wedge dy \wedge dz$$

So,

$$\int_c d\omega = \int_{I^3} d\alpha = \int_0^1 dx \int_0^1 dy \int_0^1 dz \, (\quad) = 3 \text{ terms}.$$

Look at z-term, $= \int_0^1 dx \int_0^1 dy \, [\alpha_z(x,y,1) - \alpha_z(x,y,0)]$.

We get 6 terms altogether for $\int_c d\omega$. Now consider $\int_{\partial c} \omega$.

Look at the top face of the cube: Let $g_i : I^2 \to$ top face of I^3.

$$x = u$$
$$y = v$$
$$z = 1$$

Then $(f \circ g_i)^* \omega = g_i^* f^* \omega = g_i^* \alpha$. But

$$g_i^* \alpha = \alpha_z(x,y)\, dx \wedge dy = \alpha_z(u,v)\, du \wedge dv,$$

since $dz = 0$ on top face. Thus,

$$\int_{I^2} (f \circ g_i)^* \omega = \int_{I^2} g_i^* \alpha = \int_0^1 du \int_0^1 dv\ \alpha_z(u,v,1).$$

This is one of the 6 terms from the integral $\int_c d\omega$. The other 5 add up to make $\int_c d\omega$.

Table:

homology	cohomology
$C_r(M)$	$\Omega^r(M)$
$Z_r(M)$	$Z^r(M)$
$B_r(M)$	$B^r(M)$
$H_r(M)$	$H^r(M)$.

These spaces are dual to each other, in a certain sense. Notation, let $\omega \in \Omega^r(M)$, $c \in C_r(M)$, then write

$$\int_c \omega = (\omega, c) \in \mathbb{R}.$$

Thus r-forms are real-valued, linear operators on the space of r-chains, and vice versa. This suggests that maybe $\Omega^r(M)$ is the dual space to $C_r(M)$,

$$\Omega^r(M) = C_r(M)^*.$$

These are ∞-dimensional vector spaces, so making this interpretation precise involves a big effort. We will just proceed as if it is true.

In an earlier HW problem, had vector space V, its dual V^*, a subspace $U \subset V$, and $X^* \subset V^*$, where X^* is set of forms that annihilate U. Then we had a thm,

$$\dim U + \dim X^* = \dim V. \qquad (\ker X^* = U)$$

So if U is big (high dimensionality), X^* is small and vice versa. (You can specify a vector subspace $U \subset V$ either by vectors that span it, or by forms that annihilate it.)

So what are the forms in $\Omega^r(M)$ that annihilate $Z_r(M) \subset C_r(M)$? (note if they annihilate Z_r, they annihilate B_r, too). Answer: $B^r(M)$ (coboundaries, or exact forms). How to see: Let $\beta \in B^r(M)$, $z \subset Z_r(M)$ [β = exact, z = a cycle]. Then ($\beta = d\gamma$, some $\gamma \in \Omega^{r-1}(M)$).

$$\int_z \beta = \int_z d\gamma = \int_{\partial z} \gamma = 0.$$

And what are the forms that annihilate $B_r(M) \subset Z_r(M) \subset C_r(M)$?
Ans: $Z^r(M)$ (cocycles, or closed forms). How to see: Let $b \in B_r(M)$ (a boundary, so $b = \partial c$, some $c \in C^{r+1}(M)$), and let $\zeta \in Z^r(M)$, (a closed form, $d\zeta = 0$). Then

$$\int_b \zeta = \int_{\partial c} \zeta = \int_c d\zeta = 0.$$

Conversely, interpreting $C_r(M)$ as the operators and $\Omega^r(M)$ as the operands, then $B_r(M)$ is the space that annihilates $Z^r(M)$, and $Z_r(M)$ annihilates $B^r(M)$.

What is the space dual to $H_r(M)$ (homology group)?
An element of $H_r(M)$ is $[z]$ where $z \in Z_r(M)$ is a cycle and $[z] = [z+b]$ where $b \in B_r(M)$ is a boundary. So, an operator acting on $H_r(M)$ would be one that acts on $Z_r(M)$ but annihilates boundaries, so the answer does not depend on which cycle z in $[z]$ is chosen. This means it should be a cocycle, because if $\zeta \in Z^r(M)$, then

$$(\zeta, z+b) = (\zeta, z) + \overset{0}{\cancel{(\zeta, b)}}.$$

So, $\zeta \in Z^r(M)$ can be associated with an element of $H_r(M)^*$. (You can think of $(\zeta, [z])$.) However, this element of $H_r(M)^*$ is not uniquely specified by ζ, because $\zeta' = \zeta + \beta$, where $\beta \in B^r(M)$, $\beta = d\gamma$, specifies the same map: $H_r(M) \to \mathbb{R}$:

$$(\zeta + \beta, z) = (\zeta, z) + \underbrace{(\beta, z)}_{\longrightarrow\ (d\gamma, z) = 0}.$$

Thus, the element of $H_r(M)^*$ is specified by an equivalence class $[\xi] = [\xi + \beta]$, $\beta = d\gamma$, that is, an element of $H^r(M)$. This suggests that

$$H_r(M)^* = H^r(M).$$

<u>de Rham's Theorem</u> asserts that this is correct, and moreover that in the case M is compact, $H^r(M)$ is finite dimensional. This dimensionality is the r-th Betti number,

$$b_r = \dim H_r(M) = \dim H^r(M).$$

$H^r(M)$ is properly called the <u>r-th de Rham cohomology</u> group.

<u>Remark</u>: In Stokes thm,

$$(\omega, \partial c) = (d\omega, c)$$

we can see that d is the <u>pull-back</u> of ∂. That is,

$$\partial_r : C_r(M) \to C_{r-1}(M)$$

$$\partial_r^* : C_{r-1}(M)^* \to C_r(M)^*$$

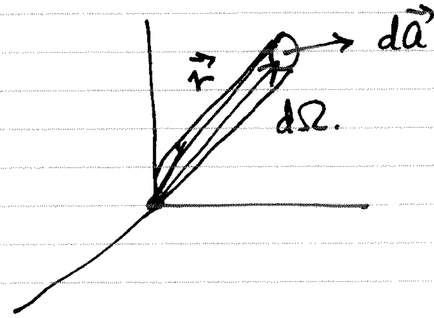
$$: \Omega^{r-1}(M) \to \Omega^r(M)$$

Thus $d_{r-1} = \partial_r^*$. Nakahara mistakenly calls this the <u>adjoint</u> (which requires a metric).

An example that will illustrate how topological information is contained in differential forms. Consider a magnetic monopole,

$$\vec{B} = g \frac{\hat{r}}{r^2}.$$

Magnetic fields are closely associated with 2-forms, because you integrate $\vec{B}$ over 2D surfaces to get fluxes. Let $d\vec{a}$ be an area element subtending solid angle $d\Omega$ at the origin,



Then by geometry,

$$\hat{r}\cdot d\vec{a} = r^2 d\Omega,$$

hence $\vec{B}\cdot d\vec{a} = g\,d\Omega$. To put this in the language of diff. forms, write β instead of $\vec{B}\cdot d\vec{a}$, and write $d\Omega$ in spherical coordinates:

$$\beta = g \sin\theta\, d\theta \wedge d\phi.$$

Since θ and ϕ are not continuous everywhere, it's not obvious that this is a smooth 2-form. So transform to Cartesian coordinates. You find

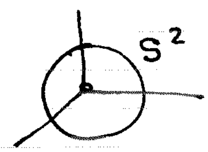
$$\beta = g \frac{x\, dy\wedge dz + y\, dz\wedge dx + z\, dx\wedge dy}{r^3},$$

which is obviously smooth everywhere except $r=0$, the location of the monopole. So to work with smooth fields, define

$$M = \mathbb{R}^3 - \{0\}.$$

Then $\beta \in \Omega^2(M)$. In fact, $d\beta = 0$, so $\beta \in Z^2(M)$.

(you can compute $d\beta$ directly.) Since $d\beta = 0$, $[\beta]$ is an element of $H^2(M)$. It is a nontrivial element (nonzero) of $H^2(M)$, because β is not <u>exact</u>. To see this, consider the integral of β over the sphere S^2 surrounding the origin:



$$\int_{S^2} \beta = 4\pi g,$$

as follows since $\beta = g\, d\Omega$. But if β were exact, $\beta = d\alpha$, then

$$\int_{S^2} \beta = \int_{S^2} d\alpha = \int_{\partial S^2} \alpha = 0$$

since S^2 is a cycle ($\partial S^2 = 0$). So β is not exact, and $[\beta] \neq 0$ defines an element of $H^2(M)$.

In ordinary language, $\beta = d\alpha$ (if it were true) would mean $\vec{B} = \nabla \times \vec{A}$. Since $\beta \neq d\alpha$, however, it means that there does not exist any $\vec{A}$ on M such that $\vec{B} = \nabla \times \vec{A}$. At least, not any <u>smooth</u> $\vec{A}$. In books it is common to introduce a "monopole string", a line on which $\vec{A}$ is singular. With such singularities, you can have $\vec{A}$ such that $\vec{B} = \nabla \times \vec{A}$.

Since $[\beta] \neq 0$, we see that $H^2(M)$ is not trivial (it is not $\{0\}$). In fact, $H^2(M)$ is one-dimensional, and it is spanned by $[\beta]$. $H^2(M) \cong \mathbb{R}$, and every element of $H^2(M)$ can be written as $a[\beta]$, where $a \in \mathbb{R}$. Equivalently, if $\omega \in Z^2(M)$, then ω can be written,

$$\omega = a\beta + d\psi$$

for some $\psi \in \Omega^1(M)$. Interpreting ω as a magnetic field flux 2-form, we see that every smooth $\vec{B}$ on M is the sum of a monopole field (of some strength) plus the curl of a smooth vector potential.

BTW, since $H^2(M) \cong \mathbb{R}$, by de Rham's thm we must have $H_2(M) = \mathbb{R}$, and there must be 2-cycles that are not boundaries. Indeed there are: S^2 is one such.

Now we develop cohomology theory and its relation to topology. First, a special case, $r=0$. We have

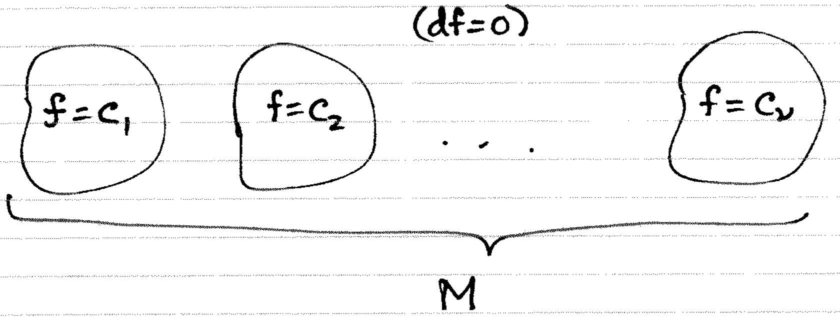
$$H^0(M) = \frac{Z^0(M)}{B^0(M)}.$$

0-forms on M are scalar fields, $f: M \to \mathbb{R}$. An "exact 0-form" is a scalar field f such that $f = d\alpha$ where α is a "(-1)-form". But (-1)-forms don't exist, so we understand that exact 0-forms don't exist either, except for $f=0$. That is, $B^0(M) = \{0\}$, so

$$H^0(M) = Z^0(M).$$

A closed 0-form is one satisfying $df=0$. This means $f=$ const. on each connected component of M. If M has ν connected components, then a closed 0-form on M is specified by its const. values on each component.

(df=0)



$\nu = \#$ connec. components.

Thus, $H^0(M) = Z^0(M) \cong \mathbb{R}^\nu$. This is the same conclusion reached earlier with homology groups, giving us an instance of de Rham's theorem.

Now let's consider <u>bases</u> in $H_r(M)$, $H^r(M)$. Start with $H_r(M)$. Let us choose a basis in $H_r(M)$. Each basis vector is an equivalence class of r-cycles, so choose one representative element in each class, call it e_i, so that the basis in $H_r(M)$ is $\{[e_i]\}$ and $e_i \in Z_r(M)$. So an arbitrary element of $H_r(M)$ can be written

$$[z] = \sum_i c_i [e_i]$$

where $z \in Z_r(M)$ and $c_i \in \mathbb{R}$ (the expansion coefficients). This means that

$$z = \sum_i c_i e_i + \partial c,$$

where $c \in C_{r+1}(M)$; thus, any cycle $z \in Z_r(M)$ can be written in this form.

Similarly for $H^r(M)$. Choose a basis in $H^r(M)$. Each basis vector is an equiv. class of closed r-forms on M. Pick representative elements, call them $\{\theta_i\}$, so that $H^r(M)$ is spanned by $\{[\theta_i]\}$, and $\theta_i \in Z^r(M)$, $d\theta_i = 0$. Now let ω be any element of $Z^r(M)$, i.e. $d\omega = 0$. Then $[\omega] \in H^r(M)$ and

$$[\omega] = \sum_i a_i [\theta_i]$$

where $a_i \in \mathbb{R}$. This means

$$\boxed{\omega = \sum_i a_i \theta_i + d\psi}$$

for some $\psi \in \Omega^{r-1}(M)$. This is the general form for a closed r-form on M.

So far this is a fairly trivial statement of the defn. of a basis and of the quotient spaces $H_r(M)$ and $H^r(M)$. Now add de Rham's theorem. It tells us that the i-sums above run from 1 to $b_r(M)$ (the Betti number). It also tells us, that since $H^r(M) = H_r(M)^*$, we can choose the basis $\{[\theta_i]\}$ in $H^r(M)$ to be dual to the basis $\{[e_i]\}$ in $H_r(M)$. Let's do that, so that

$$\big([\theta_i],[e_j]\big) = (\theta_i, e_j) = \int_{e_j} \theta_i = \delta_{ij}.$$

Then the coefficients a_i in the expansion of ω above can be computed by integrating ω over the basis cycles $\{e_i\}$. To see this just substitute,

$$(\omega, e_i) = \sum_j a_j\,(\theta_j, e_i) + \overbrace{(d\psi, e_i)}^{0} \quad \text{because } d\psi \text{ exact, } e_i \text{ a cycle}$$

$$= \sum_j a_j\,\delta_{ji} = a_i.$$

This leads to a <u>theorem</u>:

A closed form $\omega \in Z^r(M)$ is exact iff $(\omega, e_i) = \int_{e_i} \omega = 0$ for all $\{e_i\}$ in a basis in $H_r(M)$.

<u>Example of basis</u>: In monopole example above, let's take S^2 to be the basis 2-cycle, $e_1 = S^2$. Then θ_1 can be taken to be $\beta/4\pi g$, so that $(\theta_1, e_1) = 1$.

Now for formal properties of cohomology groups. Begin with the ring of differential forms on M. This is the direct sum of all forms of all possible ranks,

$$\Omega(M) = \text{ring of diff. forms on } M$$

$$= \Omega^0(M) \oplus \ldots \oplus \Omega^m(M), \qquad \text{where } m = \dim M.$$

An element of this ring is a (formal) linear combination of forms of different ranks, e.g., $dx\wedge dy + dz$. You can't integrate such over surfaces (that requires a single rank), but such formal linear combinations are useful nonetheless. One reason for defining such an object is to obtain a set that is closed under the exterior product $\wedge$. This is the definition of a <u>ring</u>, it's a set of objects that you can add or multiply, with the usual rules of distribution of addition over multiplication (and some other axioms). To say that $\Omega(M)$ is a ring mainly conveys the idea that there is a multiplication law defined, $\wedge$ in this case. A single space like $\Omega^r(M)$ is not a ring because it's not closed under $\wedge$.

Similarly we can define the <u>cohomology ring</u>,

$$H^*(M) = H^0(M) \oplus \dots \oplus H^m(M). = \text{cohomology ring}.$$

Here the multiplication law is again $\wedge$, but now defined on equivalence classes of closed forms (we have to make this definition). The obvious definition is (for $\omega \in Z^r(M)$, $\phi \in Z^s(M)$):

$$[\omega] \wedge [\phi] = [\omega \wedge \phi],$$

but we must check this for consistency. First, note that $\omega \wedge \phi$ is closed,

$$d(\omega \wedge \phi) = d\omega \wedge \phi + (-1)^r \omega \wedge d\phi = 0$$

since $d\omega = d\phi = 0$. Thus, $[\omega \wedge \phi]$ is meaningful as an element of $H^{r+s}(M)$. Next, must show that the defn is indep. of the representative element in the equivalence class. Let $\omega' = \omega + d\psi$. Then

$$[\omega' \wedge \phi] = [\omega \wedge \phi + d\psi \wedge \phi].$$

But $d\psi \wedge \phi$ is exact,

$$d(\psi \wedge \phi) = d\psi \wedge \phi + (-1)^{r-1} \underbrace{\cancel{\psi \wedge d\phi}}_{0 \text{ bec. } d\phi = 0.}$$

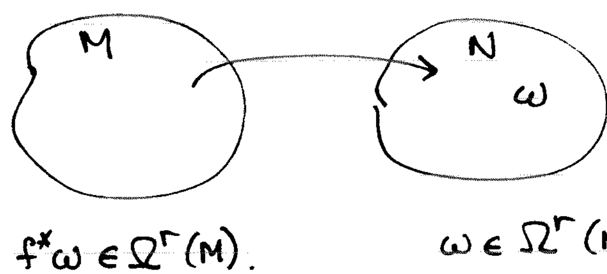
so
$$[\omega' \wedge \phi] = [\omega \wedge \phi + \text{exact}] = [\omega \wedge \phi].$$

Similarly if you write $\phi' = \phi + d\chi$. Altogether, we have shown that

$$\boxed{[\omega] \wedge [\eta] = [\omega \wedge \eta]}$$

is a consistent definition, and hence $H^*(M)$ is a ring under $\wedge$.

Now let's study behavior of cohomology groups and rings under maps. Let $f : M \to N$ be a smooth map between manifolds. We know how to pull back forms, i.e., if $\omega \in \Omega^r(N)$ then $f^*\omega \in \Omega^r(M)$.



What about cohomology group elements? Can we pull them back? Now let $\omega \in Z^r(M)$, $d\omega = 0$, and let's try to define $f^*[\omega]$ by

$$\boxed{f^*[\omega] = [f^*\omega]}$$

(the obvious defn). But we must check consistency. First, $d(f^*\omega) = f^*(d\omega) = 0$ since $d\omega = 0$, so $[f^*\omega]$ is meaningful as an element of $H^r(M)$. Next, if $\omega' = \omega + d\psi$, then

$$f^*[\omega'] = [f^*(\omega + d\psi)] = [f^*\omega + f^*d\psi] = [f^*\omega + d(f^*\psi)]$$
$$= [f^*\omega].$$

So the result is indep. of the rep. element chosen in $[\omega]$, and the definition works. We have defined a new meaning to f^*:

$$f^*: \Omega^r(N) \to \Omega^r(M) \qquad \text{(old meaning)}$$

$$f^*: H^r(N) \to H^r(M) \qquad \text{(new meaning)}.$$

f^* is a linear map of cohomology groups.

f^* also preserves the $\wedge$ between cohomology group elements, as we see by pursuing the defn. of $\wedge$ and f^* on such things:

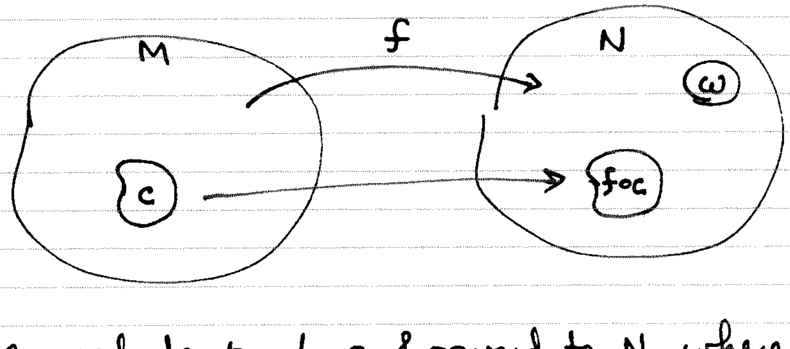
$$f^*([\omega]\wedge[\eta]) = f^*([\omega\wedge\eta]) = [f^*(\omega\wedge\eta)]$$

$$= [(f^*\omega)\wedge(f^*\eta)] = [f^*\omega]\wedge[f^*\eta]$$

$$= (f^*[\omega])\wedge(f^*[\eta]).$$

So, you can take $\wedge$ first and then apply f^*, or do it in the reverse order, answers are the same. This means that $f^*: H^*(N) \to H^*(M)$ is a ring homomorphism (another way of stating same thing).

Will need the following result concerning the behavior of integrals under maps in the subsequent discussion of potentials. Let $f: M \to N$ be a map between manifolds, let $c \in C_r(M)$ be an r-chain on M, let $\omega \in \Omega^r(N)$:



The map f can be used to push c forward to N, where it becomes