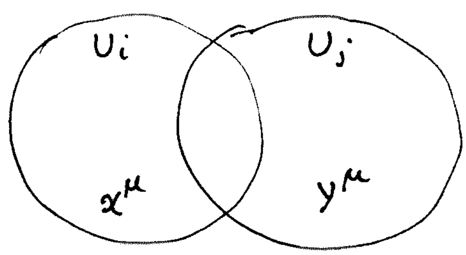
these lines:

1. Show that the Cartan formula works for $r=0$ and $r=1$.
2. Show that $i_X d + d i_X$ is a <u>derivation</u> (obeys Leibnitz) when acting on $\wedge$ products.

These steps are straightforward. They prove the Cartan formula because an arbitrary form can be represented as a linear combination of $\wedge$ products of 1-forms and 0-forms.

Now an introduction to the integration of differential forms. General idea is that r-forms get integrated over r-dimensional submanifolds of M. (Actually, the objects that they get integrated over are more general than submanifolds, they are <u>chains</u>. More about that later.) For now we concentrate on a special case, integrating m-forms on an m-dimensional manifold.

First, integration over a manifold is not meaningful unless the manifold is orientable. Consider 2 charts that overlap. $(m = \dim M)$.

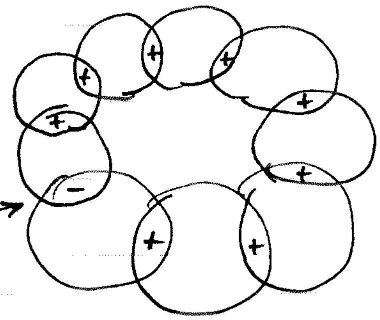


$\varphi_i : U_i \to \mathbb{R}^m$

In the overlap region, the Jacobian $\dfrac{\partial x^\mu}{\partial y^\nu}$ is nonsingular, so $\det\left(\dfrac{\partial x^\mu}{\partial y^\nu}\right)$ is either >0 or <0. If it is >0, then we say the two charts have the same orientation. Is it possible to cover M with charts that have the same orientation? Depends on M. Those manifolds that can be covered with charts that have the same orientation are said to be orientable.

For some manifolds, however, this cannot be done.

oops



Example of Möbius strip, $\mathbb{RP}^2$

There is a relation between orientability and m-forms on M. Let $\phi \in \Omega^m(M)$

Then

$$\phi = \frac{1}{m!} \phi_{\mu_1 \ldots \mu_m}(x)\, dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_m}$$

$$= \underbrace{\phi_{12\ldots m}(x)} \; dx^1 \wedge \ldots \wedge dx^m$$

↳ Call this $\rho(x)$, it is the one indep component of ϕ w.r.t. chart x^μ.

change charts, $x^\mu \to y^\mu$. Then

$$dx^\mu = \frac{\partial x^\mu}{\partial y^\nu}\, dy^\nu .$$

$$dx^1 \wedge \ldots \wedge dx^m = \det\left(\frac{\partial x}{\partial y}\right) dy^1 \wedge \ldots \wedge dy^m,$$ because of antisymmetry.

So, under the change of coordinate $x^\mu \to y^\mu$, we find $\rho \longrightarrow \rho \det\left(\frac{\partial x}{\partial y}\right)$. We say that ρ transforms as a *pseudo-scalar*. The value of ρ is not preserved under a change of coordinates, but if x^μ and y^μ have the same orientation, then the sign of ρ is conserved.

Related to a <u>thm</u>: An m-form ϕ exists on M that is nonzero everywhere iff M is orientable. This is easily proved using partitions of unity, discussed in book. Note that ϕ vanishes at a point $x \in M$ iff $\rho(x) = 0$.

If M is orientable, then we can construct atlases in which all the charts have the same orientation. Call these "oriented atlases". If we have two atlases, their charts are all either oriented the same or oriented oppositely (if they overlap). Thus the space of oriented atlases consists of two equivalence classes. We may call one of these "positively oriented" and the other "negatively oriented", but this is just a convention, not an absolute designation.

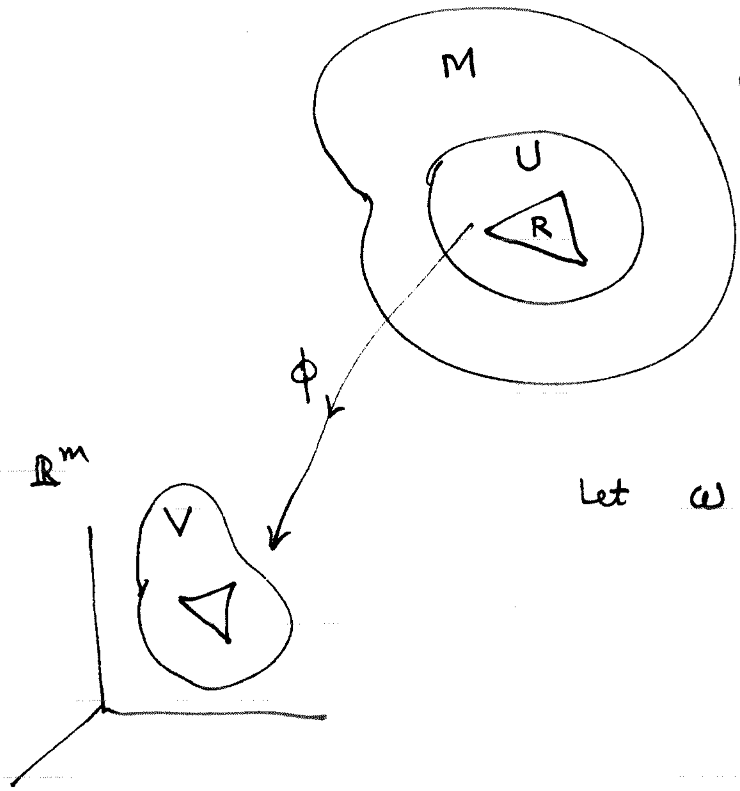
In order to specify an orientation, we choose an oriented atlas. Actually it suffices to choose a single chart, or even just a basis $\{e_\mu\}$ in a single tangent space at a single point, since if M is orientable this will fix the orientations of all other charts.

Let $\omega \in \Omega^m(M)$, and suppose we have an oriented atlas chosen.

~~Then we triangulate M so that each~~ Then to integrate ω over a region contained in one chart x^μ we define

$$\int_R \omega = \int dx^1 \ldots dx^m$$

Let $\phi : U \to V \subset \mathbb{R}^m$ be a chart containing a region R ($R \subset U$) over which we wish to integrate ω.



$\dim M = m$

$V = \phi(U)$

Let $\omega = \rho(x)\, dx^1 \wedge \ldots \wedge dx^m$

Then we define

$$\int_R \omega = \int_{\phi(R)} \rho(x)\ dx^1 \ldots dx^m .$$

The final integral is a normal Riemann integral (in particular, the answer does not depend on the ordering of the dx's.) By breaking M up into regions such that each region lies in one chart, we can ~~sum~~ add the integrals up to get $\int_M \omega$. The answer is independent of the oriented atlas we choose, apart from sign. Note that the answer is independent of the coordinates you choose, as long as the orientation is the same.

We will take the following approach to integrating differential forms:

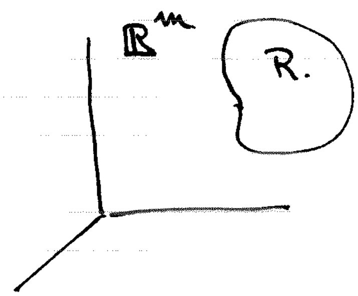
(1) Integrating an m-form over an m-dimensional region of $\mathbb{R}^m$.

(2) Integrating an m-form over an m-dimensional, orientable manifold M.

(3) Integrating an s-form over an s-dimensional orientable submanifold of M $(s \leq m)$.

(4) Integrating an r-form over an r-chain.

<u>Step 1</u>. Let $\omega \in \Omega^m(\mathbb{R}^m)$, and let R be a "nice" region of $\mathbb{R}^m$.



Then ω has only one independent component ρ, given by

$$\omega = \rho(x^1, \ldots, x^m)\, dx^1 \wedge \ldots \wedge dx^m.$$

That is, we use the standard coordinates on $\mathbb{R}^m$ to define ρ.
Then we define

$$\int_R \omega = \int_R \rho(x^1, \ldots, x^m)\, dx^1 dx^2 \ldots dx^m$$

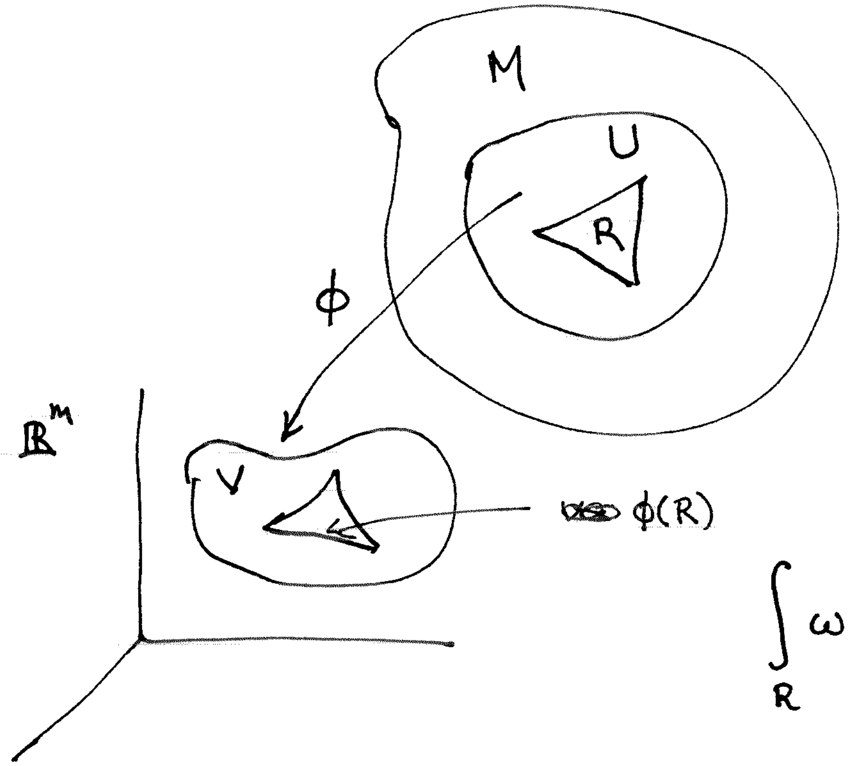
where the latter integral is an ordinary Riemann integral

(in particular, the integral does not depend on the ordering of the dx's.)

Step 2. Let M be an orientable, m-dimensional manifold, and let $\omega \in \Omega^m(M)$. We choose an oriented atlas on M, and divide M into regions $\{R_i\}$ such that each region R_i lies in one chart with coordinates x^μ. Then we define

$$\int_M \omega = \sum_i \int_{R_i} \omega,$$

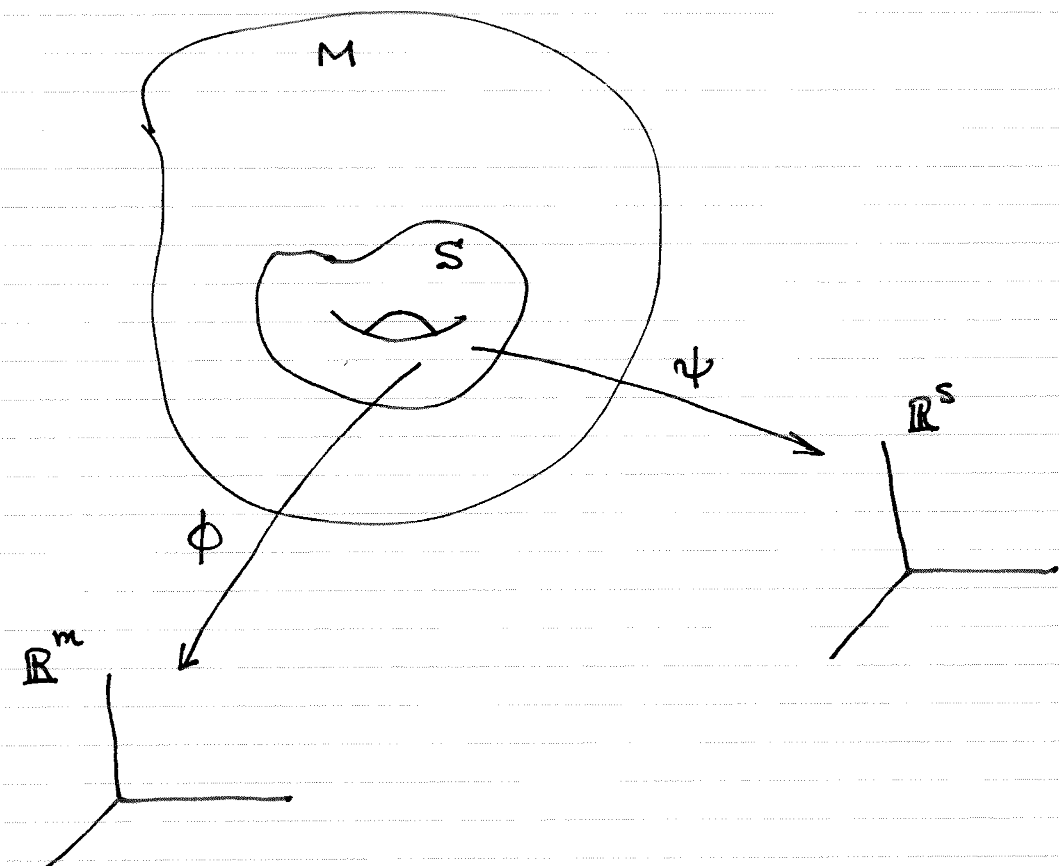
where the integral over one region R is given by



$$\int_R \omega = \int_{\phi(R)} \phi^{*-1}\,\omega.$$

The latter formula is the integral of an m-form over $\mathbb{R}^m$, which was defined in Step 1. (ϕ is the invertible coordinate map, $\phi: U \to V \subset \mathbb{R}^m$.)

Step 3. Let S be an orientable, s-dimensional submanifold of an m-dimensional manifold M. ($s \le m$). A submanifold of a manifold is a subset that is also a manifold. Let $\omega \in \Omega^s(M)$. Note that ω has $\binom{m}{s}$ indep. components in some chart.



$u^\alpha =$

Since S is a manifold, we impose coordinates $(u^1, \ldots, u^s)$ on it with some chart ψ. We let this chart have some overlap with the chart ϕ on M, with coordinates $x^\mu = (x^1, \ldots, x^m)$. In ordinary language, we would say that the functions,

$$x^\mu = x^\mu(u^\alpha) = x^\mu(u^1, \ldots, u^s)$$

are the "equations of the surface". They represent the map

$$\phi \circ \psi^{-1} : \left(\begin{smallmatrix}\text{region of}\\ \mathbb{R}^s\end{smallmatrix}\right) \to \mathbb{R}^m.$$

Now $\omega \in \Omega^s(M)$. But a form on a ~~space~~ manifold can always be restricted to a submanifold. In the present case, we denote the restricted form $\omega|_S$. It acts on tangent vectors to S at a point $x \in S$ by

$$(\omega|_S)|_x (X_1, \ldots, X_s) = \omega|_x (X_1, \ldots, X_s),$$

where $X_1, \ldots, X_s \in T_x S \subset T_x M$. The vectors $X_1, \ldots, X_s$ have a dual interpretation, as vectors tangent to M, and as tangent to S. This can also be written,

$$\omega|_S = i^* \omega,$$

where $i: S \to M$ is the <u>inclusion map</u> (identity map on S regarded as subset of M).

Notice that it is not possible, in general, to restrict vectors, only forms.

Finally, we define

$$\int_S \omega = \int_S \omega|_S,$$

which reduces ~~case~~ Step 3 to Step 2.

Note: the final integral gets reduced (as in Step 2) to integrals over the s-dimensional coordinate space. It is interesting to write one of these integrals out in terms of the components $\omega_{\mu_1 \ldots \mu_s}$ of ω on M (in chart x^μ on M).

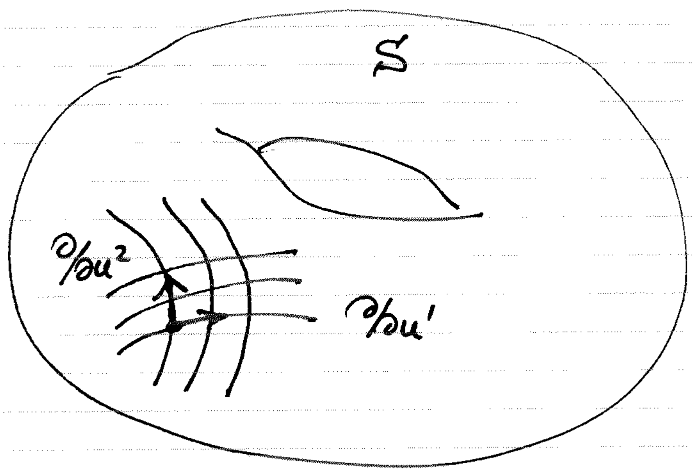
The integral has the form,

$$\int du^1 \ldots du^s \quad \underbrace{\omega_{\mu_1 \ldots \mu_s}(x(u)) \frac{\partial x^{\mu_1}}{\partial u^1} \ldots \frac{\partial x^{\mu_s}}{\partial u^s}} .$$

The integrand ( ) can also be written,

$$\omega\left(\frac{\partial}{\partial u^1}, \ldots, \frac{\partial}{\partial u^s}\right)$$

which is ω acting on the set of basis vectors on the submanifold S induced by the coordinates u^α.

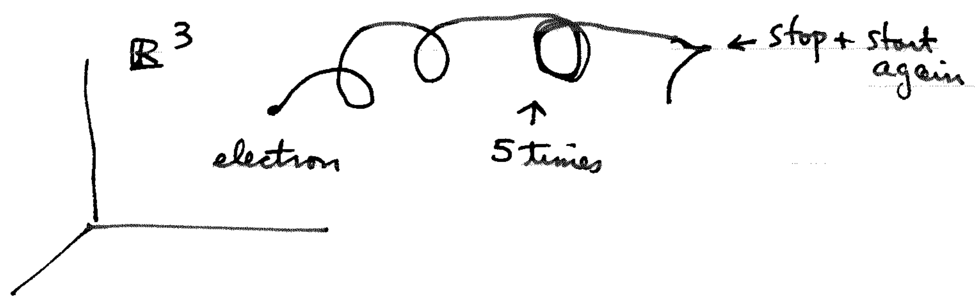


In effect, these basis vectors $\frac{\partial}{\partial u^1}, \ldots, \frac{\partial}{\partial u^s}$ span and define a small s-dimensional parallelopiped on the surface S. ω acts on this parallelepiped, producing a small contribution to the integral. The integral is the sum over the small parallelepipeds.

In Step 2 and Step 3, we require M or S to be orientable, because otherwise the integral depends on the coordinates used. If M or S are orientable, then the answer does not depend on the coordinates, apart from orientation.

that is, two atlases of the same orientation give the same ~~sign~~ answer, one of the opposite orientation reverses the sign of the answer.

Step 4. Even in simple examples, it is easy to see that integrating over manifolds or submanifolds is not sufficient for real problems. Consider for example the work required to move an election from one place to another in an electric field. This is a line integral (one-dimensional, using 1-forms). The path of the election need not be a submanifold (one-dimensional). It may have self-intersections, the path may retrace or repeat itself, or the election may stop for a while.

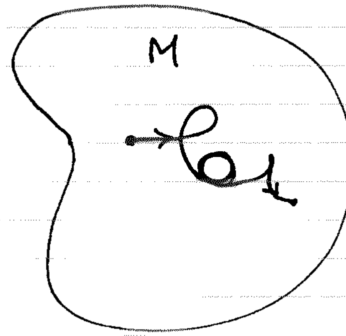


Obviously we want to parameterize the path by time or some other parameter, say, on the interval $I = [0,1]$. Thus the path is a map,

$$f : I \to M \qquad (= \mathbb{R}^3 \text{ for elector}).$$

and it is this map that we want to integrate over. The map f need not be injective and f_* need not have maximal rank.

We have seen that integrating r-forms over r-dimensional submanifolds is not general enough. For example, with $r=1$, we need to integrate over paths, that is, functions



$$f : I \to M$$

$$I = [0,1] = \text{standard region} \subset \mathbb{R}$$

If $\omega = \omega_\mu(x)\,dx^\mu$ is a 1-form on M (in chart x^μ on M), then the integral we want is

$$\int_{f:I\to M} \omega = \int_0^1 dt \quad \omega_\mu(x(t))\,\frac{dx^\mu}{dt} = \int_I f^*\omega$$

where the last integral is that of a 1-form over a region of $\mathbb{R}^1$, defined previously (Step 1 above). More generally, let us call a smooth map

$$\sigma : I^r \to M$$

a <u>singular r-cube</u>. $I^r \subset \mathbb{R}^r$ is the <u>r-cube</u>, a standard region in $\mathbb{R}^r$; the word "singular" is added to talk about the map σ, which need not be injective, nor need σ_* have maximal rank. For example, $\mathrm{img}\,\sigma$ (a subset of M) need not have dimension r, it may have self-intersections, etc. It need not be an r-dimensional submanifold of M.

Some authors (e.g. Nakahara) prefer to use a different standard region in $\mathbb{R}^r$, such as a <u>simplex</u>. Then the map is referred to as a <u>singular simplex</u>. There is no loss of generality in using cubes.

We now define the integral of an r-form $\omega \in \Omega^r(M)$ over a singular r-cube σ. It is   Here $\sigma : I^r \to M$.

$$\int_\sigma \omega = \int_{I^r} \sigma^* \omega$$

which reduces the integral to the integral of an r-form over an r-dimensional region of $\mathbb{R}^r$. To put this in coordinates, let x^μ be cooordinates on M ($\mu = 1, \ldots, m = \dim M$, $m \geq r$), and let u^α, $\alpha = 1, \ldots, r$ be the standard (Euclidean) coordinates on $\mathbb{R}^r$. Then

$$\int_\sigma \omega = \int_0^1 du^1 \ldots \int_0^1 du^r \quad \omega_{\mu_1 \ldots \mu_r}(x(u)) \frac{\partial x^{\mu_1}}{\partial u^1} \ldots \frac{\partial x^{\mu_r}}{\partial u^r}.$$

The most general integral is taken over linear combinations of singular r-cubes. We consider only real coefficients here. If $\{\sigma_i^r\}$ is a set of singular r-cubes, then we define

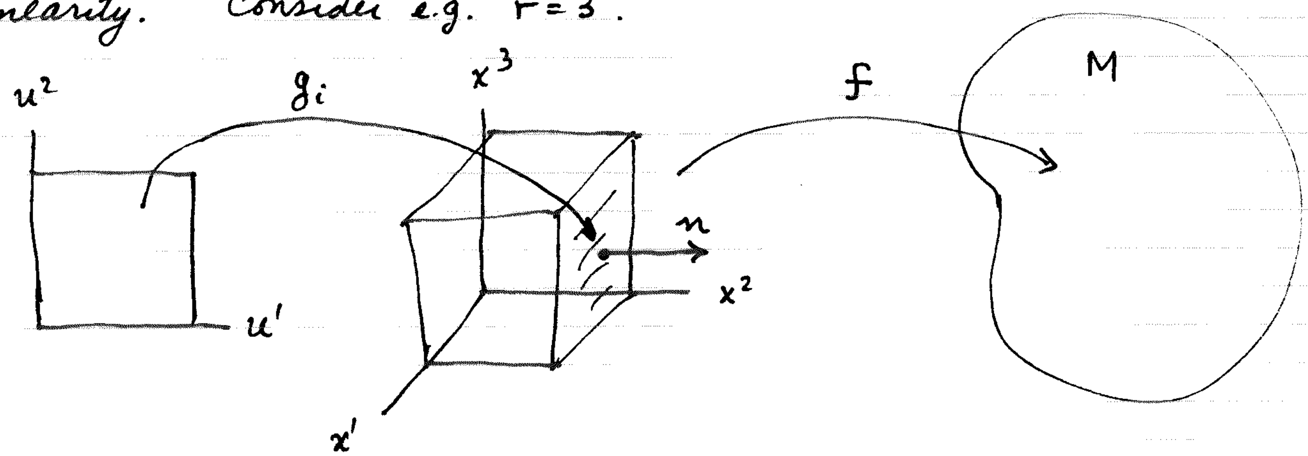
$$c^r = \sum_i a_i \sigma_i^r, \qquad a_i \in \mathbb{R}$$

as an <u>r-chain</u>. Integrals over r-chains are computed by

$$\int_{c^r} \omega = \sum_i a_i \int_{\sigma_i^r} \omega .$$

The set of all r-chains on M is the <u>r-th chain group</u>, $C_r(M, \mathbb{R})$ (we will drop the $\mathbb{R}$, it being henceforth understood.) The r-th chain group is a group in the sense that it is a vector space (an Abelian group). This is like the simplicial chain groups considered earlier, except now they include singular cubes, and now they are ∞-dimensional.

We now define the boundary operator ∂ when acting on singular r-cubes. Once that is defined, ∂ becomes defined on chains by linearity. Consider e.g. $r = 3$.



We have 6 faces, $i = 1, \dots, 6$. Each face will be associated with a singular 2-cube. But a singular 2-cube is a map from the std 2-cube (square) in $\mathbb{R}^2$ to M, and the faces of $I^3 \subset \mathbb{R}^3$ are subsets of $\mathbb{R}^3$, not $\mathbb{R}^2$. So we introduce new maps $g_i : \mathbb{R}^2 \to \mathbb{R}^3$ that map $I^2 \subset \mathbb{R}^2$ onto a face of $I^3 \subset \mathbb{R}^3$. Let x^μ be coords in $\mathbb{R}^3$ and u^α be coords in $\mathbb{R}^2$ (or (x,y,z) and (u,v)). The map g_i must assign the right orientation to the face, defined by saying that $(n, \partial/\partial u^1, \partial/\partial u^2)$ are positively oriented in $\mathbb{R}^3$, where n is an outward normal to the face, and $\partial/\partial u^1, \partial/\partial u^2$ span the face. For example, in the diagram above, we map I^2 onto the face of I^3 by writing,

$$\left.\begin{aligned} u &= z \\ v &= x \\ 1 &= y \end{aligned}\right\} .$$