Last topic is the <u>adjoint representation</u> (confusingly called the "adjoint map" by Nakahara). This is a linear action of G on its own Lie algebra, $g \mapsto Ad_g$, where $Ad_g : \mathfrak{g} \to \mathfrak{g}$. The definition is simply ~~Ad_g~~

$$Ad_a = I_{a*}\big|_e \quad \text{eval. at } e,$$

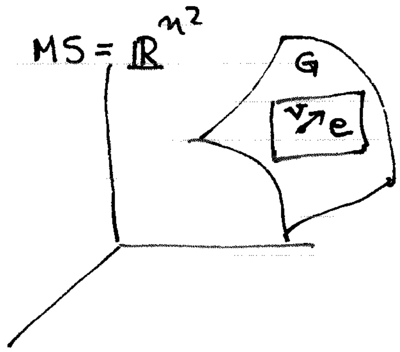
where $I_a = L_a R_{a^{-1}}$ (the inner automorphism). Thus, $I_{a*} = L_{a*} R_{a^{-1}*} = R_{a^{-1}*} L_{a*}$ (since left and right translations commute). When I_{a*} acts on a vector $V \in T_e G = \mathfrak{g}$, first L_{a*} maps it to a vector in $T_a G$, then $R_{a^{-1}*}$ maps it back to $\mathfrak{g}$. So, I_{a*} ~~action~~ maps $\mathfrak{g} \to \mathfrak{g}$. It also satisfies $I_{a*} I_{b*} = (I_a I_b)_* = I_{ab*}$, since $a \mapsto I_a$ is an action. Thus, (changing notation),

$$Ad_a Ad_b = Ad_{ab}.$$

For a matrix group, a vector $V \in \mathfrak{g}$ is represented by a matrix V, group element a is rep'd by a matrix A, and $Ad_a V$ is rep'd by the matrix AVA^{-1}. This is the adjoint rep. for matrix groups.

(Go to for notes on integrating n-forms over an n-dimensional manifold.)

Now we consider how things like the Lie algebra, one-parameter subgroups, etc., are expressed in terms of matrices in the case that we have a matrix group. Consider a real matrix group, for simplicity. As explained previously, such a group can be thought of as a submanifold of "matrix space" $MS = \mathbb{R}^{n^2}$, where our group consists of real, $n \times n$ matrices.



Then every point of G is also a matrix, and matrix multiplication and inversion ~~f~~ correspond to group multiplication and inversion. (and $e = I$)

We will not attempt to put coordinates on G, but coordinates on MS may be taken to be the components of a matrix. That is, if $M \in MS$, let us write

$$M = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix}$$

so that $\{x_{ij}\}$ are the coordinates on MS.

A vector at a point to G can also be interpreted as a vector at the same point to MS, and thus can be expanded as a linear combination of the basis vectors $\partial/\partial x_{ij}$. For example, if $V \in \mathfrak{g}$, then we can write

$$V = \sum_{ij} V_{ij} \frac{\partial}{\partial x_{ij}},$$

where V_{ij} is a matrix. This is the usual matrix belonging to the Lie algebra of a matrix group. For example, if $G = SO(3)$, $MS = \mathbb{R}^9$, then the Lie algebra consists of antisymmetric matrices. Then, $V_{ij} = -V_{ji}$.

Then it also happens that the one-parameter subgroups $\exp(tV)$ defined in the differential-geometric setting coincide with matrix exponentiation $\exp(tV)$ (same notation). It also happens that the $[\,,]$ bracket on the Lie algebra becomes the ordinary matrix commutator. Other objects (left and right translations, left-invariant vector fields, etc.) can also be translated into matrix language.

Return to the differential geometry of Lie groups. Let $\{V_\mu, \mu = 1, \ldots, n\}$ ($n = \dim V$) be a basis in $\mathfrak{g}$. Let $X_\mu = X_{V_\mu}$ be the corresponding LIVF's. Now the bracket $[V_\mu, V_\nu]$ is also a vector in $\mathfrak{g}$, so it can be expanded in terms of the $\{V_\mu\}$,

$$[V_\mu, V_\nu] = C_{\mu\nu}{}^{\sigma}\, V_\sigma,$$

where $C_{\mu\nu}{}^{\sigma}$ are the expansion coefficients. These numbers are called the <u>structure constants</u> of the Lie algebra, although they are not really constant, instead they are the components of a type $(1,2)$ tensor at $e \in G$. (They depend on the basis.) If we left-translate the above, we get

$$[X_\mu, X_\nu] = C_{\mu\nu}{}^{\sigma}\, X_\sigma,$$

with the same $C_{\mu\nu}{}^{\sigma}$ (which do not depend on position).

A different point of view results from shifting attention from <u>vector fields</u> to <u>forms</u> (the dual point of view). Let $\mathfrak{g}^* = T_e^* G$ be the dual of $\mathfrak{g}$ (the Lie algebra). Let $\{\beta^\mu, \mu = 1, \ldots, n\}$ be the basis in $\mathfrak{g}^*$ dual to $\{V_\mu, \mu = 1, \ldots, n\}$, the (some) given basis in $\mathfrak{g}$.

That is, $\beta^\mu \in \mathfrak{g}^*$,

$$\beta^\mu(V_\nu) = \delta^\mu_\nu.$$

Then define a 1-form $\theta^\mu \in \mathcal{X}^*(G)$ by

$$\theta^\mu\big|_a = L^*_{a^{-1}}\, \beta^\mu.$$

(The difference betw. β^μ and θ^μ is that β^μ is a covector at one point $e \in G$, whereas θ^μ is a covector field, i.e., a 1-form. It is like the difference between $V_\mu \in \mathfrak{g}$ and $X_\mu \in \mathcal{X}(G)$.) The forms θ^μ are <u>left-invariant 1-forms</u> on G. The set $\{\theta^\mu\}$ is dual to $\{X_\mu\}$ at each point $a \in G$, as we see by using the definitions,

$$\theta^\mu(X_\nu)\Big|_a = \theta^\mu\big|_a\left(X_\nu\big|_a\right) = \left(L^*_{a^{-1}}\beta^\mu\right)\left(L_{a*} V_\nu\right)$$

$$= \beta^\mu\left(L_{a^{-1}*}\, L_{a*}\, V_\nu\right) = \beta^\mu(V_\nu) = \delta^\mu_\nu.$$

Thus we have bases $\{X_\mu\}$ and $\{\theta^\mu\}$ ~~at each~~ of vectors and 1-forms at each point of G. These are generally non-coordinate bases (see HW). It is of interest to compute the components of $d\theta^\mu$ in this basis.

$$(d\theta^\mu)(X_\nu, X_\sigma) = \underbrace{X_\nu\, \theta^\mu(X_\sigma)}_{= X_\nu\, \delta^\mu_\sigma = 0} - \underbrace{X_\sigma\, \theta^\mu(X_\nu)}_{\text{also} = 0} - \theta^\mu([X_\nu, X_\sigma])$$

$$= -\theta^\mu\left(C_{\nu\sigma}{}^\tau X_\tau\right) = -C_{\nu\sigma}{}^\mu.$$

So the structure constants (with a $-$ sign) are the components of $d\theta^\mu$ in the basis of LIVF's $\{X_\mu\}$. The 2-form $d\theta^\mu$ (in the abstract, for a fixed value of μ) is

$$\boxed{d\theta^\mu = -\tfrac{1}{2} C_{\nu\sigma}{}^{\mu}\, \theta^\nu \wedge \theta^\sigma.}$$

Maurer-Cartan structure equations.

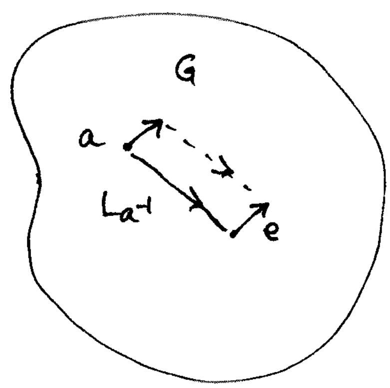
To put things in completely coordinate independent language, we write

$$\theta = V_\mu \otimes \theta^\mu.$$

θ is an example of a <u>Lie-algebra valued 1-form</u>. So far we have only seen real-valued 1-forms, ~~that is,~~ but Lie-alg. valued 1-forms are important in gauge theories (gauge potentials are such things).
θ is a map, (at a point $a \in G$)

$$\theta|_a : T_a G \to \mathfrak{g}.$$

It is easy to see abstractly what θ does: it uses left translation to map a vector in $T_a G$ to one in $T_e G = \mathfrak{g}$.



θ is called the <u>Maurer-Cartan form</u>. One might say that every Lie group carries on itself a gauge potential. The MC form can be written in fully coordinate-free notation if we define

$$d\theta = d(V_\mu \otimes \theta^\mu) = V_\mu \otimes d\theta^\mu,$$

logical since the V_μ are constant. This makes $d\theta$ a Lie-algebra-valued

2-form. Also define

$$[\theta,\theta] = [V_\mu, V_\nu]\otimes\theta^\mu\wedge\theta^\nu,$$

another $\mathfrak{g}$-valued 2-form. Then the MC structure equations can be written,

$$\boxed{d\theta + \frac{1}{2}[\theta,\theta] = 0.}$$

Note, in QCD you get Lie-algebra valued 1-forms, this is the gauge potential, A^a_μ where $\mu = 0,..,3$ is a space-time index and $a = 1,\dots,8$ is an index of the basis in the $SU(3)$ Lie algebra, e.g., the index of the Gell-Mann matrices. Call these V_a. Then

$$V_a\, A^a_\mu\, dx^\mu$$

is a $\mathfrak{g}$-valued 1-form on space-time. (~~And~~

(And, $F = \frac{1}{2} V_a\, F^a_{\mu\nu}\, dx^\mu\wedge dx^\nu$ is the Yang-Mills field tensor.)