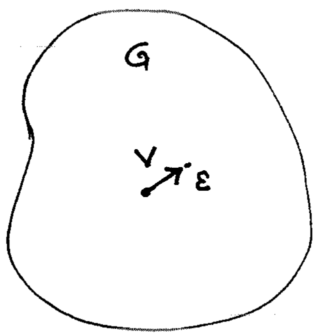
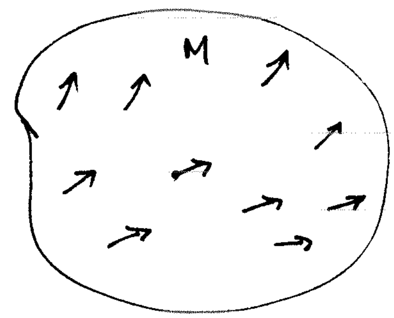
Next we consider <u>induced vector fields</u>, which you have when you have an action of a Lie group G on a manifold M. First the intuitive picture. Consider a vector $V \in \mathfrak{g}$. Intuitively its base is the identity e and its tip is a nearby (near-identity) group element, call it ε. The map $\Phi_e = \mathrm{id}_M$ does



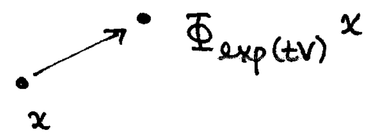


a single vector

nothing to points of M, but Φ_ε causes the points of M to get up and move a short distance, creating a vector field on M. In this way we associate $V \in \mathfrak{g}$ with a vector field $V_M \in \mathfrak{X}(M)$. (V_M is denoted $V^\#$ by Nakahara.)

~~To make this more precise, replace ε by $\sigma(t) = e = \exp(tV)$ for small t, using earlier notation for integral curves on the group manifold, and consider the action of Φ~~

To make this more precise we need to talk about advance maps on the group manifold, earlier denoted $\Phi_{V,t}$, and the action of G on M, which will be denoted $g \mapsto \Phi_g$. To avoid confusion, let's use $\Psi_{V,t}$ for the advance map on G, Φ_g for the action of G on M. When t is small, $\Psi_{V,t}\, e = \exp(tV)$ is close to e, so we can identify it with ε above in the picture. When acting on $x \in M$, $\Phi_\varepsilon = \Phi_{\exp(tV)}$ causes x to move to a nearby point,



thereby making a small vector on M. Letting this vector act on a ~~function~~ scalar $f : M \to \mathbb{R}$, we have

$$f\left(\Phi_{\exp(tV)}\, x\right) - f(x)$$

$$= \left(\Phi^*_{\exp(tV)}\, f\right)(x) - f(x)$$

$$= \left(\left(\Phi^*_{\exp(tV)} - 1\right) f\right)(x). \qquad \text{(think } t \text{ small).}$$

Suggests we define $V_M \in \mathfrak{X}(M)$ by

$$(V_M f)(x) = \left(\left(\frac{d}{dt}\Big|_{t=0} \Phi^*_{\exp(tV)}\right) f\right)(x),$$

or, since x and f are arbitrary,

$$\boxed{V_M = \frac{d}{dt}\Big|_{t=0} \Phi^*_{\exp(tV)}}$$

(Both sides understood as operators: $\mathcal{F}(M) \to \mathcal{F}(M)$.)

See Nakahara Eq. 5.160. He drops the star on Φ and just writes g instead of Φ_g, where here $g = \exp(tV)$.

V_M is called the <u>induced vector field</u>. It is also called the <u>infinitesimal generator</u> of the action $g \mapsto \Phi_g$.

An equivalent way to define the induced vector field. V_M eval. at a point $x \in M$ is an equivalence class of curves. One of these curves is easy to write down. Let $c : \mathbb{R} \to M$ be defined by

$$c(t) = \Phi_{\exp(tV)}\, x.$$

Then $c(0) = x$, and $[c] = V_M|_x$.

An application of induced vector fields. Let M be the configuration space of a mechanical system. Impose a chart with coordinates q^μ. The Lagrangian is a function $L(q, \dot{q})$. Let G be a group with an action $g \mapsto \Phi_g$ on M, and suppose that L is invariant under the group action. This means that $\Phi_g^* L = L$, $\forall\, g \in G$. (But we won't define what Φ_g^* means here, just say that there is an obvious definition.) Then for every $V \in \mathfrak{g}$ there is a conserved quantity C_V, where

$$C_V = (p, V_M) = p_\mu (V_M)^\mu, \qquad \frac{dC_V}{dt} = 0.$$

Here $p_\mu = \dfrac{\partial L}{\partial \dot{q}^\mu}$ is the canonical momentum. This is <u>Noether's theorem</u>.

Now an application of induced vector fields and an intro to geometric mechanics, namely, <u>Noether's theorem</u>. Let M be the configuration manifold of a mechanical system, and let x^i be coordinates on M. They need not be rectangular coordinates. The Lagrangian is a function $L(x^i, \dot{x}^i)$. It is not a function on M, but rather on TM. A point of TM is a tangent vector v attached to a point $x \in M$. If x^i are the coordinates of x and $v = \dot{x}^i \partial/\partial x^i$, then $x^i, \dot{x}^i$ are coordinates on TM, and the Lagrangian is usually expressed in terms of these coordinates. Let's henceforth use v^i instead of $\dot{x}^i$ for coordinates on the fiber T_xM, since it's less confusing. Thus

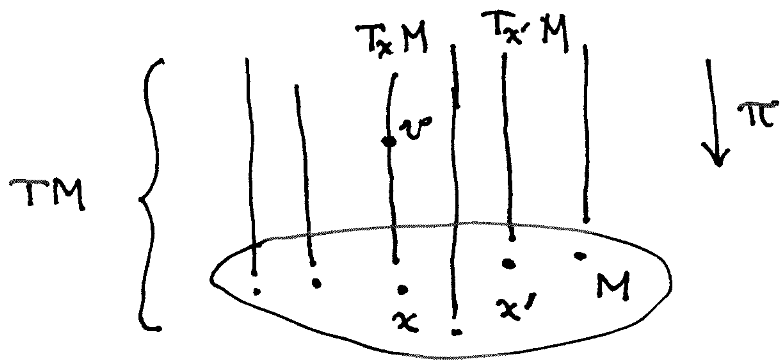
(or rather, we'll use $\dot{x}^i$ or v^i, whichever seems less confusing.)

$$v = v^i \partial/\partial x^i$$

and

$$L : TM \to \mathbb{R}.$$

Here's a picture of the tangent bundle and some fibers:



The bundle has the usual projection,

$$\pi : TM \to M : (x, v) \mapsto x,$$

which just throws away the velocity information. Notice that

$$\pi^{-1}x = T_xM = \text{the fiber over } x.$$

Now suppose we have a Lie group G with an action $g \mapsto \Phi_g$ on M, so that $\Phi_g \Phi_h = \Phi_{gh}$, and $\Phi_g : M \to M$. Let $V \in \mathfrak{g}$, and let $\exp(\lambda V)$ be the corresponding 1-parameter subgroup of G, with λ as a parameter. Then the corresponding induced vector field on M is (Since we will use t for "real" time.)

$$V_M = \left.\frac{d}{d\lambda}\right|_{\lambda=0} \Phi^*_{\exp(\lambda V)} \quad \text{acting on } \mathcal{F}(M),$$

or as an equivalence class of curves through $x \in M$,

$$V_M|_x = [\Phi_{\exp(\lambda V)}\, x],$$

i.e., $c(\lambda) = \Phi_{\exp(\lambda V)}\, x$, $c : \mathbb{R} \to M$.

Noether's theorem applies when the Lagrangian is invariant under the group action (this is the simplest version of the theorem). But the Lagrangian is not a function on M, rather it's on TM, so we have to say what this means. This is easy in coordinates. Let

$$y = \Phi_{\exp(\lambda V)}\, x,$$

(Here y^i are the coordinates of point y, and x^i of point x, both in the same coordinate system.)

or, in coordinates,

$$y^i = \Phi^i(\lambda, x) \quad (V \text{ understood}). \qquad \text{(A)}$$

Then we will say that L is invariant under the group action if

$$L(x^i, \dot{x}^i) = L(y^i, \dot{y}^i), \qquad \text{(B)}$$

where y^i is a function of x and λ as in (A), and where $\dot{y}^i$ means

$$\dot{y}^i = \frac{\partial y^i}{\partial x^j}\dot{x}^j = \frac{\partial \Phi^i(\lambda, x)}{\partial x^j}\dot{x}^j \qquad \text{(C)}$$

Eq. (c) is the obvious application of the chain rule to (A). ~~Eq.~~
Note that the induced vector field has components,

$$(V_M)^i = \left.\frac{\partial \Phi^i}{\partial \lambda}\right|_{\lambda=0}, \tag{D}$$

because that is the rate of change of the coordinates w.r.t. λ when the group action $\Phi_{\exp(\lambda V)}$ is applied to $x \in M$. Now Eq. (B) is supposed to hold for all λ, but the LHS doesn't depend on λ, so if we apply $d/d\lambda$ we get

$$0 = \frac{\partial L}{\partial y^i}\frac{\partial \Phi^i}{\partial \lambda} + \frac{\partial L}{\partial \dot{y}^i}\frac{\partial^2 \Phi^i}{\partial \lambda \partial x_j}\dot{x}^j .$$

Now setting $\lambda = 0$, where $x^i = y^i$, $\dot{x}^i = \dot{y}^i$, and (D) holds, we get

$$0 = \frac{\partial L}{\partial x^i} V_M^i + \frac{\partial L}{\partial \dot{x}^i}\frac{\partial V_M^i}{\partial x^j}\dot{x}^j .$$

But by Euler Lagrange eqns,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) = \frac{\partial L}{\partial x^i},$$

this gives

$$0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) V_M^i + \frac{\partial L}{\partial \dot{x}^i}\frac{d}{dt}(V_M^i)$$

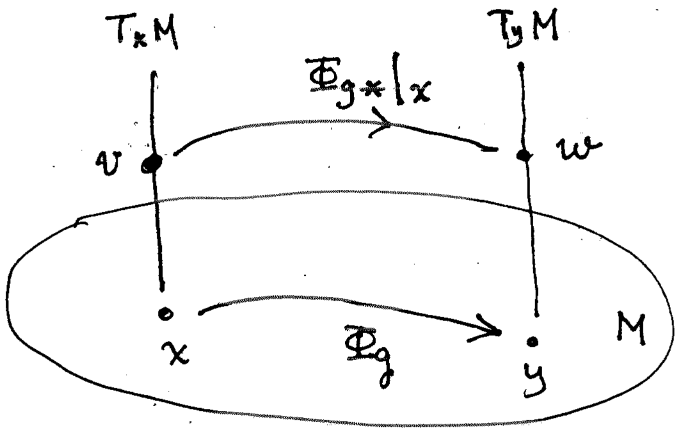
$$= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i} V_M^i\right) = 0.$$

The quantity $\partial L/\partial \dot{x}^i \equiv p_i$ is the <u>momentum</u> conjugate to x^i, so we have

$$\frac{d}{dt}\left(p_i V_M^i\right) = 0. \tag{E}$$

The conserved quantity is the momentum contracted with the the induced vector field of the group action. This is Noether's theorem. It provides a map from $\mathfrak{g}$ to conserved quantities, that is scalars on TM that are invariant under the time evolution.

Now for some geometrical interpretation. We started with an action of G on M, $g \mapsto \Phi_g$, $\Phi_g : M \to M$. But to talk about the invariance of L under the group action, we need an action of G on TM. The key formula is (C), which shows how the velocity should transform. The Jacobian $\partial y^i/\partial x^j$ in this formula shows that the new velocity $\dot{y}^i$ is the tangent map of Φ_g applied to the old velocity. To draw a picture, let $y = \Phi_g x$ and let $v \in T_x M$.



Let $v = \dot{x}^i \, \partial/\partial x^i|_x$ and $w = \dot{y}^i \, \partial/\partial x^i|_y$. Then $w = \Phi_{g*}|_x v$ is equivalent to (C). What we have obtained is a <u>lifted</u> action of G on TM, call it $g \mapsto \tilde{\Phi}_g$, $\tilde{\Phi}_g : TM \to TM$, associated with Φ_g on M.

$$\tilde{\Phi}_g(x, v) = (\Phi_g x, \Phi_{g*}|_e v),$$

where $v \in T_x M$. Then (B) is equivalent to

$$L(x, v) = L(\tilde{\Phi}_g(x, v)) = (\tilde{\Phi}_g^* L)(x, v),$$

or

$$\tilde{\Phi}_g^* L = L.$$

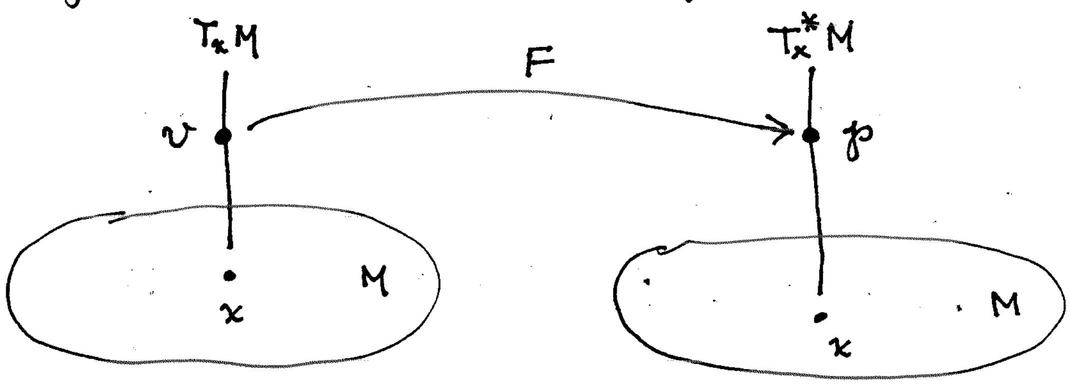
The Lagrangian is invariant under the lifted action. If we now ~~apply~~ set $g = \exp(\lambda V)$ and apply $\frac{d}{dt}\Big|_{\lambda=0}$, we get

$$\mathcal{L}_{V_{TM}} L = 0,$$

where V_{TM} is the induced vector field on TM assoc. with the action $\tilde{\Phi}_{\exp(tV)}$, and $\mathcal{L}$ is the Lie derivative.

The final result (E) for the conserved qty is obviously the contraction of a 1-form with components p_i with the induced vector field V_M of the symmetry. The momentum appears as the components of a 1-form. But it is not a field of forms over M, that is, it is not a member of $\Omega^1(M)$, because $p_i = \partial L/\partial \dot{x}^i$ is a function of both x^i and $\dot{x}^i$. That it, it is an association of each point of TM with a specific covector. This is usually regarded as a map from the tangent bundle to the cotangent bundle.

$F : TM \to T^*M$
$: (x,v) \to (x,p)$



where

$p_i = \dfrac{\partial L}{\partial \dot{x}^i}(x,\dot{x})$ $\qquad v = v^i \dfrac{\partial}{\partial x^i}\Big|_x$ $\qquad p = p_i\, dx^i\big|_x$

This map F is essentially the Legendre transformation of mechanics. It preserves fibers, so it can also be thought of as a map: $T_xM \to T_x^*M$ (x does not change under the map).
for every $x \in M$.

Just looking at what $F: T_xM \to T_x^*M$ does on a single fiber, the equation specifying F is just the usual definition of the momentum,

$$p_i = \frac{\partial L}{\partial v^i}(x, v).$$

~~This~~ The map is invertible if the Jacobian

$$\frac{\partial p_i}{\partial v_j} = \frac{\partial^2 L}{\partial v_i \partial v_j}$$

has full rank everywhere, i.e., if its determinant is $\neq 0$. Lagrangians for which

$$\det\left(\frac{\partial^2 L}{\partial v_i \partial v_j}\right) \neq 0$$

everywhere on TM are said to be <u>regular</u>. For a regular Lagrangian, $F: TM \to T^*M$ is a diffeomorphism, which can be used to push-forward any geometrical structure (e.g. the time flow) from TM to T^*M, or conversely to pull anything back from T^*M to TM.

In particular, the Hamiltonian, defined as

$$H = p_i \dot{x}^i - L(x, \dot{x})$$

begins life as a function on TM, but pushed forward under F to T^*M it becomes a function of (x, p). Then one can show that the equations of motion on T^*M are Hamilton's equations,

$$\left.\begin{aligned} \dot{x}^i &= \partial H / \partial p_i \\ \dot{p}_i &= -\partial H / \partial x^i \end{aligned}\right\}.$$

This is a vector field on T^*M, that is, a section of $T(T^*M)$.

This is the beginning of geometrical mechanics, about which we will say more later.

The Lagrangians of elementary, nonrelativistic mechanics are almost always regular, but relativistic mechanics and field theory (and general relativity) lead to (usually) irregular Lagrangians. More about that later.