Another $(r=1)$ example of d. Let $A \in \Omega^1(M)$,

$$A = A_\mu \, dx^\mu$$

Then
$$dA = A_{\mu,\nu} \, dx^\nu \wedge dx^\mu$$
$$= \frac{1}{2} \underbrace{(A_{\nu,\mu} - A_{\mu,\nu})}_{\text{components of } F = dA, \quad F \in \Omega^2(M).} dx^\mu \wedge dx^\nu.$$

$$F = \frac{1}{2} F_{\mu\nu} \, dx^\mu \wedge dx^\nu .$$

Example with $r=2$. Let $F \in \Omega^2(M)$,

$$F = \frac{1}{2} F_{\mu\nu} \, dx^\mu \wedge dx^\nu$$

$$dF = \frac{1}{2} F_{\mu\nu,\sigma} \, dx^\sigma \wedge dx^\mu \wedge dx^\nu$$
$$= \frac{1}{3!} \underbrace{\left(F_{\mu\nu,\sigma} + F_{\sigma\mu,\nu} + F_{\nu\sigma,\mu} \right)}_{\text{components of } dF.} dx^\mu \wedge dx^\nu \wedge dx^\sigma$$

We recognize these examples from E+M (A_μ = vector potential, $F_{\mu\nu}$ = field tensor).

Properties of d.

(1) Distributive law on $\wedge$ product. Let $\alpha \in \Omega^r(M)$, $\beta \in \Omega^s(M)$.

Then $d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^r \alpha \wedge (d\beta)$.

(2) If $\alpha \in \Omega^r(M)$, then

$$d\alpha(X_1, \ldots, X_{r+1}) = \sum_{i=1}^{r+1} (-1)^{i+1} X_i \, \alpha(X_1, \ldots, \overset{\overset{\text{omit.}}{\downarrow}}{\hat{X}_i}, \ldots, X_{r+1})$$
$$+ \sum_{i<j} (-1)^{i+j} \alpha([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{r+1}).$$

(3) $d^2 = 0$.

Comments. Property (2) is equivalent to defn of d (since it gives action of $d\alpha$ on arb. set of vectors). Turns out properties (1)+(3) also imply defn of d. Prop. (1) follows easily from "$d = \partial\wedge$" (you use a chain rule, but in order to bring the d in to act on β in the 2nd term, you must commute it through α, which introduces $(-1)^r$ factor.)

Proof of property (3):

$$\alpha = \frac{1}{r!}\,\alpha_{\mu_1\ldots\mu_r}\, dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_r}$$

$$d\alpha = \frac{1}{r!}\,\alpha_{\mu_1\ldots\mu_r,\nu}\, dx^{\nu}\wedge dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_r} \qquad \text{(defn. of d).}$$

$$dd\alpha = \frac{1}{r!}\,\alpha_{\mu_1\ldots\mu_r,\underbrace{\nu\sigma}_{\text{symm}}}\, \underbrace{dx^{\sigma}\wedge dx^{\nu}}_{\text{antisymm}}\wedge\ldots\wedge dx^{\mu_r} = 0.$$

Special cases of (2):

$r=0$, $f \in \mathcal{F}(M)$.

$$df(X) = Xf.$$

$r=1$, $\alpha \in \Omega^1(M)$

$$d\alpha(X,Y) = X\alpha(Y) - Y\alpha(X) - \alpha([X,Y])$$

$r=2$, $\beta \in \Omega^2(M)$

$$d\beta(X,Y,Z) = X\beta(Y,Z) - Y\beta(X,Z) + Z\beta(X,Y)$$
$$- \beta([X,Y],Z) + \beta([X,Z],Y) - \beta([Y,Z],X).$$

(4) Another property of d. Let $f: M \to N$ be a map (not nec. a diffeo.). Let $\alpha \in \Omega^r(N)$, so $f^*\alpha \in \Omega^r(M)$. Then

$$\boxed{f^*(d\alpha) = d(f^*\alpha)} \qquad \text{d commutes with pull-backs.}$$

Easy to prove in components/coordinates.

Important terminology.

An r-form $\alpha \in \Omega^r(M)$ is closed if $d\alpha = 0$.

It is exact if $\exists\ \beta \in \Omega^{r-1}(M)$ such that $\alpha = d\beta$.

Now we consider the interior product. Let $X \in \mathfrak{X}(M)$, then the interior product is an operator $i_X : \Omega^r(M) \to \Omega^{r-1}(M)$, defined by ($\alpha \in \Omega^r(M)$):

$$(i_X\alpha)(Y_1, \dots, Y_{r-1}) = \alpha(X, Y_1, \dots, Y_{r-1}).$$

This is a purely algebraic operation (just insert X into 1st ~~to~~ slot of α), no differentiation required. Notice that i_X lowers the rank of α, while d raises it. Properties of i_X:

(1) $\quad i_X(\alpha \wedge \beta) = (i_X\alpha)\wedge\beta + (-1)^r \alpha \wedge (i_X\beta), \qquad \alpha \in \Omega^r(M)$

(2) $\quad i_X^2 = 0$

(3) $\quad \boxed{\mathcal{L}_X = i_X d + d i_X} \qquad$ (Acting on forms).

Property (3) is the Cartan formula. The geometrical meaning of this formula must wait until we cover Stokes' theorem. A proof is given in the book. The proof I prefer runs along these lines:

① Show that the Cartan formula works for $r=0$ and $r=1$.

② Show that $i_X d + d i_X$ is a <u>derivation</u> (obeys Leibnitz) when acting on $\wedge$ products.

These steps are straightforward. They prove the Cartan formula because an arbitrary form can be represented as a linear combination of $\wedge$ products of 0-forms and 1-forms.