Then let $\vec{J}$ be a flux vector (of mass, charge, etc., or maybe $\vec{J}=\vec{B}$ = magnetic field). Then the flux through parallelogram is

$$\vec{J}\cdot(\vec{\xi}\times\vec{\eta}).$$

Why $\vec{\xi}\times\vec{\eta}$ and not $\vec{\eta}\times\vec{\xi}$? Because we have to decide which side of the parallelogram is the "outward" oriented side (it's a convention, but the sign of the answer depends on it). So the area element is specified by $\vec{\xi}\times\vec{\eta}$, which is antisymmetric in the two vectors. And the value of the flux is the value of a real-valued linear operator that acts on area elements. It's like a covector (acts on vectors), except that it acts on 2 vectors (effectively area elements). Note, we can write

$$\vec{J}\cdot(\vec{\xi}\times\vec{\eta}) = \frac{1}{2}J_{ij}\,(\xi^i\eta^j - \xi^j\eta^i)$$

where $J_{ij} = \varepsilon_{ijk}J^k$. $J_{ij} = -J_{ji}$ are the components of a 2-form.

Special cases of r-forms:

$r=0$ is a scalar, or 0-form, considered to be antisymmetric in its nonexistent operand.

$r=1$ is a covector, or 1-form, considered to be antisymmetric in its one operand, $\alpha: \mathfrak{X}(M) \to \mathcal{F}(M)$ (as a field)

$r=2$ is a 2-form, an antisymmetric tensor acting on two vector fields,

$$\omega: \mathfrak{X}(M)\times\mathfrak{X}(M) \to \mathcal{F}(M),$$

$$\omega(X,Y) = -\omega(Y,X).$$

Cases $r=0,1,2$

In **components**: let $x \in M$, x^i = coordinates of x,

let $e_i = \dfrac{\partial}{\partial x^i}$ = basis vectors of coordinate system.

$= 0$-form

A scalar $f: M \to \mathbb{R}$ has ~~no~~ only one component, the value $f(x)$ of f itself.

A covector or 1-form α has components,

$$\alpha_i(x) = \alpha(e_i)\big|_x .$$

A 2-form ω has components,

$$\omega_{ij}(x) = \omega(e_i, e_j)\big|_x = -\omega_{ji}(x).$$

The number of independent components of an r-form on an n-dim'l space is

$$\binom{n}{r} = \begin{cases} 1 & r=0 \\ n & r=1 \\ \dfrac{n(n-1)}{2} & r=2 \\ \vdots & \\ 1 & r=n. \end{cases}$$

Another special case is an n-form, call it ϕ. This is a completely antisymmetric map of n vectors to scalars,

$$\phi : \mathfrak{X}(M) \times \dots \times \mathfrak{X}(M) \to \mathfrak{F}(M)$$

with components

$$\phi_{i_1 \dots i_n}(x) = \text{completely antisymmetric in indices } (i_1, \dots, i_n).$$

$$= \phi(e_{i_1}, \dots, e_{i_n})$$

Thus, a n-form on an n-dimensional ~~sp~~ manifold has components in any given chart that have the form

$$\phi_{i_1 \dots i_n}(x) = \sigma(x)\, \varepsilon_{i_1 \dots i_n},$$

where $\varepsilon_{i_1 \dots i_n}$ is the Levi-Civita symbol, and $\sigma(x)$ is a <u>scalar density</u>. $\sigma(x)$ defines the one and only indep. component of ϕ.

We write the set of all smooth r-forms on M as $\Omega^r(M)$. Thus, $\Omega^0(M) = \mathcal{F}(M)$, $\Omega^1(M) = \overset{*}{\mathfrak{X}}(M)$, etc.

We consider r-forms with $r > n$ to be zero.

How to construct r-forms. One way is to take the <u>exterior product</u> of r 1-forms. The exterior product is an <u>antisymmetrized tensor product</u>. ~~For~~ The exterior product of a r 1-forms is defined as follows.

Let $\alpha^1, \dots, \alpha^r$ be 1-forms $\quad (\alpha^i \in \Omega^1(M))$.

Then $\alpha^1 \wedge \dots \wedge \alpha^r$ is an r-form, defined by its action on r vector fields $X_1, \dots, X_r$ by

$$\underbrace{(\alpha^1 \wedge \dots \wedge \alpha^r)}_{r\text{-form}}(X_1, \dots, X_r) = \sum_{P \in \mathcal{S}_r} (-1)^P \alpha^1(X_{P_1})\, \alpha^2(X_{P_2}) \dots \alpha^r(X_{P_r})$$

where $\mathcal{S}_r$ = set of all permutations P of r objects. More precisely, P is a bijection of the set $\{1, 2, \dots, r\}$ to itself, P_i = value of P acting on i $(1 \le i \le r)$. $(-1)^P$ is the <u>parity</u> of the permutation ($+1$ if even, -1 if odd).

Can also write this as

$$(\alpha^1 \wedge \dots \wedge \alpha^r)(X_1, \dots, X_r) = \begin{vmatrix} \alpha^1(X_1) & \dots & \alpha^1(X_r) \\ \vdots & & \vdots \\ \alpha^r(X_1) & \dots & \alpha^r(X_r) \end{vmatrix}$$

<u>Example</u>: Let $\alpha, \beta \in \Omega^1(M)$, $X, Y \in \mathfrak{X}(M)$

$$(\alpha \wedge \beta)(X, Y) = \begin{vmatrix} \alpha(X) & \alpha(Y) \\ \beta(X) & \beta(Y) \end{vmatrix} = \alpha(X)\beta(Y) - \alpha(Y)\beta(X).$$

Properties:

1) $\alpha^1 \wedge \dots \wedge \alpha^r$ is completely antisymmetric,

$$\alpha^{P_1} \wedge \dots \wedge \alpha^{P_r} = (-1)^P \alpha^1 \wedge \dots \wedge \alpha^r.$$

in particular, $\alpha \wedge \beta = -\beta \wedge \alpha$ $\quad (\alpha, \beta \in \Omega^1(M))$.

2) If $\alpha^i = \alpha^j$ for any $i \neq j$, then $\alpha^1 \wedge \dots \wedge \alpha^r = 0$.

A general r-form is not the exterior product of a set of r 1-forms, but can always be represented as a linear combination of such products. <u>Example</u>: Let A be an antisymmetric, $(0,2)$ tensor,

$$A = A_{\mu\nu}\, dx^\mu \otimes dx^\nu \qquad (A_{\mu\nu} = -A_{\nu\mu})$$

$$= \tfrac{1}{2} A_{\mu\nu} \left[dx^\mu \otimes dx^\nu - dx^\nu \otimes dx^\mu \right]$$

$$= \tfrac{1}{2} A_{\mu\nu}\, dx^\mu \wedge dx^\nu .$$

Thus, $dx^\mu \wedge dx^\nu$ $(\mu,\nu = 1,\dots,n)$ is a basis of 2-forms on M.

Similarly, for a general r-form,

$$\overset{\text{r-form}}{\omega} = \frac{1}{r!}\,\omega_{\mu_1\dots\mu_r}(x)\,\underbrace{dx^{\mu_1}\wedge\dots\wedge dx^{\mu_r}}_{\text{basis of r-forms.}}$$

Here we use summation convention, $\sum_{\mu_1,\dots,\mu_r}$ implied. If you some only over indices in ascending order,

$$\sum_{\mu_1<\mu_2<\dots<\mu_r}$$

you can drop the factor of $\frac{1}{r!}$.

Now generalize the exterior product to arbitrary forms.

Let $\alpha \in \Omega^r(M)$, $\beta \in \Omega^s(M)$. Then $\alpha\wedge\beta \in \Omega^{r+s}(M)$, defined by

$$(\alpha\wedge\beta)(X_1,\dots,X_{r+s}) = \frac{1}{r!\,s!}\sum_{P\in S_{r+s}} \overset{(-1)^P}{\downarrow}\,\alpha(X_{P_1},\dots,X_{P_r})\,\beta(X_{P_{r+1}},\dots,X_{P_{r+s}}).$$

Can simplify this.

Properties:

1) $(\alpha\wedge\beta)\wedge\gamma = \alpha\wedge(\beta\wedge\gamma)$ (Associative)

2) $\alpha\wedge\beta = (-1)^{rs}\,\beta\wedge\alpha$.

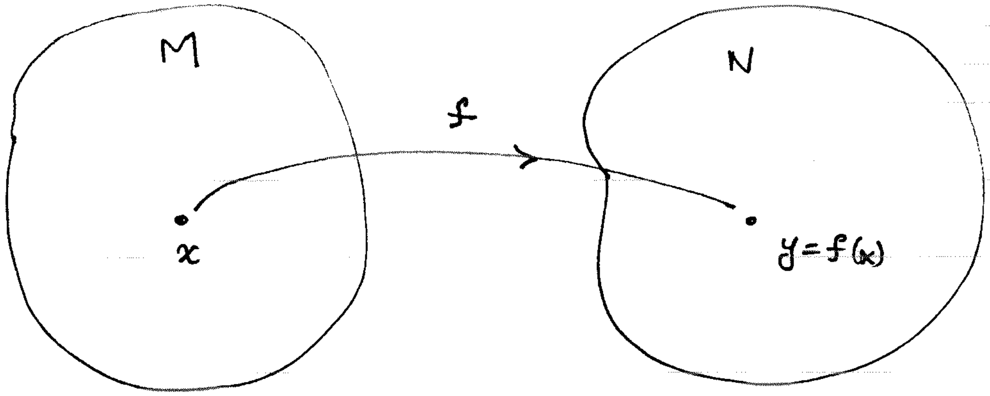
Reason: $(-1)^{rs}$ because need rs exchanges to swap order of factors.

Note: 2) implies $\alpha\wedge\alpha = 0$ when r = odd.

Note special case, $r=0$, α = 0-form $\equiv f$. Then $f\wedge\beta = f\beta$ (ord. mult.)

3) $f^*(\alpha\wedge\beta) = f^*\alpha\wedge f^*\beta$ where $f: M\to N$.

Behavior of differential forms under maps. Let $f : M \to N$ be a smooth map between two manifolds (not necessarily of same dimensionality). Let $x \in M$ and let $y = f(x) \in N$.



Let $\omega \in \Omega^r(N)$. Thus ω evaluated at $y = f(x)$, denoted $\omega|_y = \omega|_{f(x)}$, is an operator acting on r tangent vectors $\in T_y N$. Now we define $f^*\omega \in \Omega^r(M)$ by showing its action on vectors $X_1, \ldots, X_r \in T_x M$. Note that these are vectors at a point, not vector fields. Then the definition is

$$\boxed{f^*\omega\big|_x (X_1, \ldots, X_r) = \omega\big|_{f(x)} (f_* X_1, \ldots, f_* X_r).}$$

f_* = tangent map.

Doing this at each point $x \in M$ defines $f^*\omega$, the pull-back of ω under f. We previously defined the pull-back on scalars (0-forms), and on covectors (1-forms). This definition agrees with those in the cases $r = 0$ or $r = 1$.

Some comments. The exterior product of a set of 1-forms was defined by

$$(\alpha^1\wedge\dots\wedge\alpha^r)(X_1,\dots,X_r)=\begin{vmatrix}\alpha^1(X_1) & \dots & \alpha^1(X_r)\\ \vdots & & \\ \alpha^r(X_1) & \dots & \alpha^r(X_r)\end{vmatrix} = \sum_{P\in S_r}(-1)^P\,\alpha^1(X_{P_1})\dots\alpha^r(X_{P_r}).$$

Note, regarding the tensor product,

$$(\alpha^1\otimes\dots\otimes\alpha^r)(X_1,\dots,X_r) = \alpha_1(X_1)\,\alpha_2(X_2)\dots\alpha^r(X_r).$$

Therefore another definition of wedge product of r 1-forms is

$$\alpha^1\wedge\dots\wedge\alpha^r = \sum_{P\in S_r}(-1)^P\ \alpha^{P_1}\otimes\dots\otimes\alpha^{P_r}.$$

Example, $\alpha^1\wedge\alpha^2 = \alpha^1\otimes\alpha^2 - \alpha^2\otimes\alpha^1$.

Now let $\omega\in\Omega^r(M)$. Then

$$\omega = \omega_{\mu_1\dots\mu_r}\ dx^{\mu_1}\otimes\dots\otimes dx^{\mu_r}.$$

Actually, an expression like this holds for any type $(0,r)$ tensor. But, since ω is antisymmetric, we also have

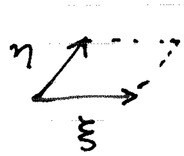
$$\boxed{\omega = \frac{1}{r!}\ \omega_{\mu_1\dots\mu_r}\ dx^{\mu_1}\wedge\dots\wedge dx^{\mu_r}.}$$

this is a standard way of writing a diff. form in terms of its components.

Here the (implied) sum is, each $\mu_i = 1,\dots,m$ $(m=\dim M)$. If you restrict to a definite order, you can drop the $\frac{1}{r!}$. Thus,

$$\omega = \sum_{\mu_1<\mu_2<\dots<\mu_r}\omega_{\mu_1\dots\mu_r}\ dx^{\mu_1}\wedge\dots\wedge dx^{\mu_r}.$$

<u>Now the exterior derivative.</u> Real motivation for this requires Stoke's theorem, for which later. For now just imagine integrating a 1-form α around a small parallelogram defined by two <u>small</u> vectors ξ, η:



$$\oint \alpha = \oint \alpha_\mu \, dx^\mu$$

Opposite sides obviously cancel to lowest order, so answer must involve derivatives of $\alpha_\mu(x)$. In fact, you find,

$$\oint \alpha = \frac{1}{2} \underbrace{(\alpha_{\nu,\mu} - \alpha_{\mu,\nu})}_{\text{antisymmetric in } \mu,\nu} (\xi^\mu \eta^\nu - \xi^\nu \eta^\mu) + \text{higher order}.$$

interpreted as components of the <u>2-form</u> $\underline{\beta = d\alpha}$.

Idea of exterior derivative: map $d : \Omega^r(M) \to \Omega^{r+1}(M)$,

$$d\alpha = \text{“} \partial \wedge \alpha \text{”}.$$

Define in coordinates. Let

$$\alpha = \frac{1}{r!} \alpha_{\mu_1 \dots \mu_r} \, dx^{\mu_1} \wedge \dots \wedge dx^{\mu_r}.$$

Then

$$d\alpha = \frac{1}{r!} \alpha_{\mu_1 \dots \mu_r, \nu} \, dx^\nu \wedge dx^{\mu_1} \wedge \dots \wedge dx^{\mu_r}.$$

<u>Examples:</u> Let $f \in \mathcal{F}(M) = \Omega^0(M)$.

$$df = f_{,\mu} \, dx^\mu.$$ This is the covector df, the "differential of a function" defined earlier.