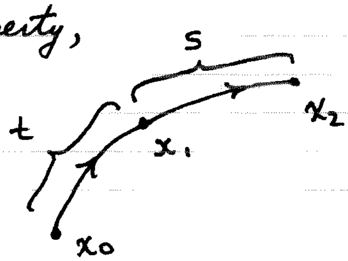
$$x = \Phi(t, x_0) = \Phi_t(x_0)$$

is the point on the integral curve starting at x_0 at $t=0$, reached after time t.

The advance map satisfies an important property,



$$\Phi_s \Phi_t = \Phi_{s+t}$$

or $$\Phi(s, \Phi(t, x_0)) = \Phi(s+t, x_0).$$

(the <u>composition property</u>). This is intuitive: if you start at x_0, follow the integral curve for elapsed time t, reaching x_1, then treat x_1 as initial conditions and follow the integral curve for elapsed time s, you must get the same thing as starting at x_0 and following the integral curve for time $s+t$.

We will prove this working in a single chart, ignoring the complications that result when we must switch charts. We want to show that

$$\Phi^i(s, \Phi(t, x_0)) = \Phi^i(s+t, x_0).$$

Let $x_1^i = \Phi^i(t, x_0)$, $\quad \xi^i(s) = \Phi^i(s, x_1)$, $\quad \eta^i(s) = \Phi^i(s+t, x_0)$.

We need to show that $\xi^i(s) = \eta^i(s)$. First, at $s=0$, we have

$$\xi^i(0) = \Phi^i(0, x_1) = x_1^i$$
$$\eta^i(0) = \Phi^i(t, x_0) = x_1^i.$$

Next, we have

$$\frac{d\xi^i}{ds} = \frac{\partial \Phi^i}{\partial s}(s, x_1) = X^i(\Phi(s, x_1)) = X^i(\xi(s)),$$

and

$$\frac{d\eta^i}{ds} = \frac{\partial \Phi^i}{\partial s}(s+t, x_0) = \frac{\partial \Phi^i}{\partial(s+t)}(s+t, x_0)$$

$$= X^i(\Phi(s+t, x_0)) = X^i(\eta(s)).$$

Thus, both $\xi^i(s)$ and $\eta^i(s)$ satisfy the same ODE's and the same initial conditions, so by the uniqueness theorem they must be equal. QED.

By the composition property, $\Phi_{-t} \circ \Phi_t = id_M$, so Φ_t is a diffeomorphism : $M \to M$. In fact, the set

$$\{\Phi_t \mid t \in \mathbb{R}\}$$

constitutes a one-parameter group of diffeomorphisms of M onto itself. It is an action of the group $\mathbb{R}$ (meaning t) on M. This group is sometimes called the flow.

Now about the exponential notation for the flow. This is a way of connecting Φ_t with the vector field X. The notation used in many books is

$$\boxed{\Phi_t = e^{tX}.}$$

Taken ~~as written~~ literally, this has no meaning (we must assign a meaning to it). First note that a vector field X is a mapping : $\mathcal{F}(M) \to \mathcal{F}(M)$, namely, $f \mapsto \sum_i X^i \frac{\partial f}{\partial x^i}$. Therefore $X^2 = X \cdot X = X \circ X$ has a meaning as a map: $\mathcal{F}(M) \to \mathcal{F}(M)$. The higher powers of a vector field are not vector fields (they are not 1st order partial differential operators, they are higher order diff. ops.), but they are perfectly good maps : $\mathcal{F}(M) \to \mathcal{F}(M)$. [Notice that a vector at a point is a map: $\mathcal{F}(M) \to \mathbb{R}$, so a power of it has no meaning.]

So, an exponential series like

$$e^{tX} = 1 + tX + \frac{t^2}{2!}X^2 + \cdots$$

has meaning as a map: $\mathcal{F}(M) \to \mathcal{F}(M)$, at least if we ignore convergence questions (which we will). The 1 above means id_M. On the other hand, Φ_t is a map: $M \to M$, not: $\mathcal{F}(M) \to \mathcal{F}(M)$. This is why $\Phi_t = e^{tX}$ has no meaning as it stands. But let us apply the exponential series to a function f and see what we get. We evaluate the function at x_0, which is an initial condition. ~~(we write x instead of x_0)~~

$$e^{tX} f = f + tXf + \frac{t^2}{2} X^2 f + \cdots$$

$$= f + t \sum_i X^i \frac{\partial}{\partial x^i} f + \frac{t^2}{2} \sum_i X^i \frac{\partial}{\partial x^i} \sum_j X^j \frac{\partial}{\partial x^j} f + \cdots$$

$$= f + t \text{ "}\frac{df}{dt}\text{" } + \frac{t^2}{2} \text{ "}\frac{d^2 f}{dt^2}\text{" } + \cdots$$

where we put the t-derivatives in quotes because what is actually meant is

$$\text{"}\frac{df}{dt}\text{"} = \frac{d}{dt}(f \circ c)$$

where c is the integral curve passing through x at $t=0$. Thus, if the series converges, it does so to "$f(t)$", which means $f(x(t))$, where $x(t)$ is the integral curve, $x(t) = \Phi_t(x_0)$. So,

$\uparrow$ means same as $c(t)$

$$(e^{tX} f)(x_0) = f(\Phi_t(x_0)) = (\Phi_t^* f)(x_0).$$

This is true for all x_0, so we have

$$e^{tX} f = \Phi_t^* f.$$

And this is true for all f, so we have

$$\Phi_t^* = e^{tX}. \qquad (A)$$

The usual formula in books is meaningful if we put a $*$ on Φ_t (turning it into a pull-back), and interpret both sides as maps : $\mathcal{F}(M) \to \mathcal{F}(M)$.

On the other hand, we can regard e^{tX} as a formal notation for Φ_t, without trying to interpret it as a power series. This notation has the virtue of making some of the properties of the advance map obvious:

$$\Phi_s \Phi_t = e^{sX} e^{tX} = e^{(s+t)X} = \Phi_{s+t}$$

$$\text{and } \Phi_0 = e^{0X} = 1 = \mathrm{id}_M.$$

Restoring the star $*$, we can differentiate $\Phi_t^* = e^{tX}$ formally, to get

$$\frac{d}{dt}\Phi_t^* = X e^{tX} = e^{tX} X,$$

which implies

$$\frac{d}{dt}\Phi_t^* = X \Phi_t^* = \Phi_t^* X, \qquad (B)$$

an equation that is perfectly meaningful as operators : $\mathcal{F}(M) \to \mathcal{F}(M)$. We have not proved it (because of questions of convergence of series), but in fact this result is true, and can be proved by other means.

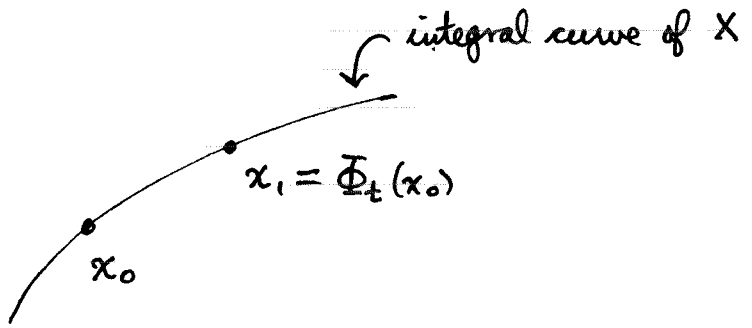
Altogether, we have derived relationships between a vector field X and the 1-parameter group of diffeomorphisms $\{\Phi_t\}$ that it generates, in an intrinsic notation (not tied to a coordinate system). These are Eqs. (A) and (B) above.

Continue today with the <u>Lie derivative</u>, which is like the convective derivative of ordinary tensor analysis, but generalized to arbitrary manifolds.

<u>Context</u>: Given a manifold M, a vector field $X \in \mathfrak{X}(M)$, with advance map $\Phi_t : M \to M$. Illustrate Lie derivative first with scalar fields, where $\mathcal{L}_X : \mathcal{F}(M) \to \mathcal{F}(M)$ ($\mathcal{L}_X$ is the Lie derivative along vector field X.)

Let $x_0 \in M$ and $x_1 = \Phi_t x_0$. We think of t as small (we will be interested in the limit $t \to 0$). For $f \in \mathcal{F}(M)$, define

$$(\mathcal{L}_X f)(x_0) = \lim_{t\to 0} \frac{1}{t}\Big[f(x_1) - f(x_0) \Big]$$



It's pretty obvious from this formula that $\mathcal{L}_X f = Xf$, since the vector $X|_{x_0}$ is the small displacement $x_0 \to x_1$ in small time t. ~~But write~~ Thus, the Lie derivative of a scalar is the obvious generalization of the convective derivative to an arbitrary manifold,

$$\mathcal{L}_X f = Xf = \sum_i X^i \frac{\partial f}{\partial x^i}. \qquad (\text{Think } \vec{v}\cdot\nabla f).$$

But transform equ. above:

$$(\mathcal{L}_X f)(x_0) = \lim_{t\to 0} \frac{1}{t}\Big[f(\Phi_t x_0) - f(x_0) \Big]$$

$$= \lim_{t = 0} \frac{1}{t}\Big[(\Phi_t^* f)(x_0) - f(x_0) \Big]$$

$$= \lim_{t\to 0} \frac{1}{t}\left(\left[\Phi_t^* - 1\right] f\right)(x_0)$$

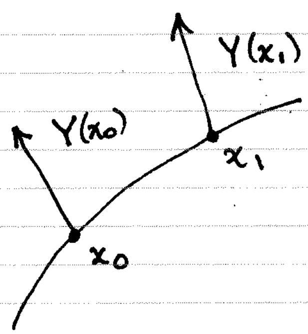
$$= \left(\left(\frac{d}{dt}\bigg|_{t=0} \Phi_t^*\right) f\right)(x_0).$$

But recall,

$$\Phi_t^* = e^{tX}$$

when acting on scalars, so we find again, $\left.\frac{d\Phi_t^*}{dt}\right|_{t=0} = X$, $\mathcal{L}_X f = Xf$.

Now generalize to other differential geometric objects, like vector fields. Now we want to define $\mathcal{L}_X$ as an operator: $\mathfrak{X}(M) \to \mathfrak{X}(M)$. Let $Y \in \mathfrak{X}(M)$ (now we have 2 vector fields, X and Y), and we ~~wt~~ wish to define $\mathcal{L}_X Y$. The idea is the same as above, we wish to compare Y at x_1 with Y at x_0 to see how much Y has changed along the integral curves of X. But we cannot just subtract



$Y(x_1) - Y(x_0)$, these vectors belong to two different tangent spaces $T_{x_0}M$ and $T_{x_1}M$ without any natural identification. ~~Not can we~~ ~~Instead, we~~ However, we can "pull-back" $Y(x_1)$ to point x_0 using the flow (mapping both base and tip of arrow by Φ_t^{-1}). Note that the pull-back of a vector field is defined in this case because Φ_t is invertible; the pull back is the inverse of the tangent map Φ_{t*}. So, we define

$$\mathcal{L}_X Y = \left(\frac{d}{dt}\Big|_{t=0} \Phi^{-1}_{t*}\right) Y, \text{ or}$$

$$(\mathcal{L}_X Y)(x_0) = \lim_{t\to 0} \frac{1}{t}\left[(\Phi^{-1}_{t*} Y)(x_0) - Y(x_0)\right]$$

In components,

$$(\Phi^{-1}_{t*} Y)^i (x_0) = \frac{\partial x_0^i}{\partial x_1^j} Y^j(x_1).$$

To get x_1 as a fn of x_0, t, we solve the ODE's,

$$\frac{dx^i}{dt} = X^i(x)$$

in power series in t,

$$x_1^i = x_0^i + t X^i(x_0) + \dots$$

or its inverse,

$$x_0^i = x_1^i - t X^i(x_0) + \dots$$

So,

$$\frac{\partial x_0^i}{\partial x_1^j} = \delta^i_j - t X^i_{,j} + \dots$$

and,

$$= \left[\delta^i_j - t X^i_{,j}\right] Y^j (x_0 + t X + \dots)$$

$$= Y^i(x_0) + t\left(X^j Y^i_{,j} - Y^j X^i_{,j}\right).$$

So,

$$\boxed{(\mathcal{L}_X Y)^i = X^j Y^i_{,j} - Y^j X^i_{,j}}$$

This is the Lie derivative of a vector field.

Similarly, you can define the Lie deriv. of a covector field

by
$$\mathcal{L}_X\alpha = \left(\frac{d}{dt}\Big|_{t=0} \Phi_t^*\right)\alpha .$$

If you work it out, you find (in components),

$$\boxed{(\mathcal{L}_X\alpha)_i = X^j \alpha_{i,j} + \alpha_j X^j{}_{,i}}$$

To define $\mathcal{L}_X$ on arbitrary tensors, we develop some general rules. First, $\mathcal{L}_X$ acts on a tensor product of tensors by the Leibnitz rule. An example will illustrate. Consider the tensor product of a covector with a vector (this is a type (1,1) tensor): Define

$$\mathcal{L}_X (\alpha \otimes Y) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \left[(\Phi_\varepsilon^* \alpha) \otimes (\Phi^{-1}_{\varepsilon *} Y) \right],$$

which is the obvious definition. But this is...

$$\boxed{(\mathcal{L}_X \alpha)\otimes Y + \alpha \otimes (\mathcal{L}_X Y).} \quad \boxed{= \mathcal{L}_X(\alpha\otimes Y)}$$

(Leibnitz rule).

The same thing works on contractions. For example, the tensor $\alpha\otimes Y$ has components,

$$(\alpha \otimes Y)_i{}^j = \alpha_i Y^j$$

If we contract (set $i=j$ and sum), we get

$$\alpha_i Y^i = \alpha(Y). = \text{a scalar.}$$

Then we have

$$\mathcal{L}_X [\alpha(Y)] = (\mathcal{L}_X\alpha)(Y) + \alpha(\mathcal{L}_X Y)$$

$$= X(\alpha(Y)).$$ Can use this to calculate $\mathcal{L}_X\alpha$ in components, supposing that we know what $\mathcal{L}_X$ does to scalars

and vector fields.

Notice that a scalar multiplied by tensor is a special case of a tensor product:

$$f \otimes T = f\,T \qquad \text{any } T, \quad f \in \mathcal{F}(M).$$

Therefore

$$\mathcal{L}_X (f T) = (\mathcal{L}_X f) T + f(\mathcal{L}_X T) = (Xf) T + f(\mathcal{L}_X T).$$

Since an arbitrary tensor field can be written as linear combinations of scalars times tensor products of vector fields and covector fields, the Leibnitz rule suffices to compute the Lie derivative of any tensor.

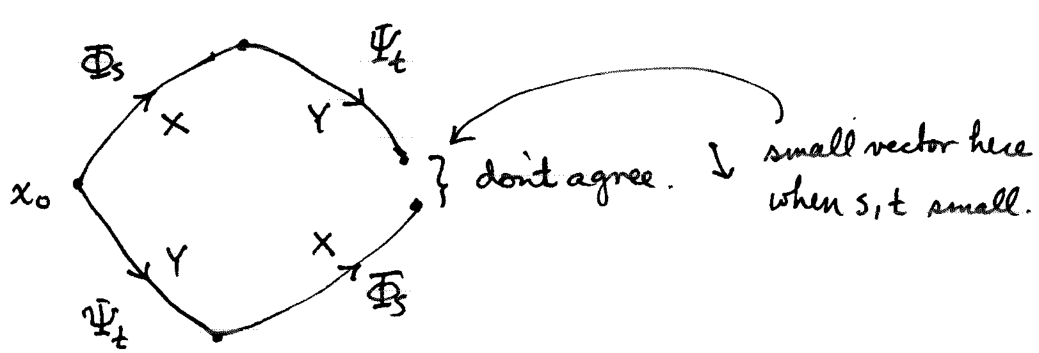
~~Some more~~ rules about $\mathcal{L}_X$. If $f \in \mathcal{F}(M)$, then

WRONG ignore

$$\boxed{\mathcal{L}_{fX} = f \mathcal{L}_X}$$

This is obvious since fX has same integral curves as X, except the t-parametrization is scaled by f. Hence $\frac{d}{dt}\Big|_{t=0}$ is scaled by f.

The Lie derivative $\mathcal{L}_X Y$ is a special case with a special interpretation. Consider the flows associated with X, Y, call them Φ_s, Ψ_t. These in general do not commute,



When s,t are small, the difference in the endpoints must be a vector. However, since we cannot subtract points, to measure the difference between $\Psi_t \Phi_s x_0$ and $\Phi_s \Psi_t x_0$ we evaluate some scalar $f: M \to \mathbb{R}$ at the 2 points and subtract:

$$f(\Psi_t \Phi_s x_0) - f(\Phi_s \Psi_t x_0)$$

$$= \left((\Psi_t \Phi_s)^* f\right)(x_0) - \left((\Phi_s \Psi_t)^* f\right)(x_0)$$

$$= (\Phi_s^* \Psi_t^* f)(x_0) - (\Psi_t^* \Phi_s^* f)(x_0)$$

$$= \Big(\big(\underbrace{\Phi_s^* \Psi_t^* - \Psi_t^* \Phi_s^*}\big) f\Big)(x_0).$$

$$= e^{sX} e^{tY} - e^{tY} e^{sX}$$

$$= \left(1 + sX + \frac{s^2}{2}X^2 + \dots\right)\left(1 + tY + \frac{t^2}{2}Y^2 + \dots\right) - (X \leftrightarrow Y)$$

$$= 1 + (sX + tY) + \left(\frac{s^2}{2}X^2 + stXY + \frac{t^2}{2}Y^2\right) + \dots - (X \leftrightarrow Y)$$

$$= st\,(XY - YX) + \dots$$

Thus the small vector is $[X, Y]$ times st.

$$= st\,[X, Y] + \dots$$

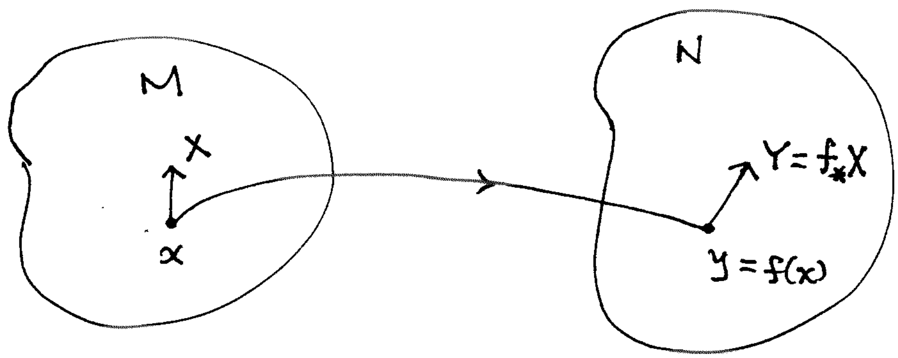
Thus the leading term is the <u>commutator</u> of X and Y (regarded as maps: $\mathcal{F}(M) \to \mathcal{F}(M)$). Thus we have

$$\left.\frac{\partial^2}{\partial s\, \partial t}\right|_{t=0} \left[f(\Psi_t \Phi_s x_0) - f(\Phi_s \Psi_t x_0)\right] = ([X, Y] f)(x_0),$$

$$\forall f \in \mathcal{F}(M).$$

A digression, regarding some points that I should have made earlier, regarding pull-backs and push-forwards of vector and covector fields, in contrast to vectors and covectors at a point. First, vectors.

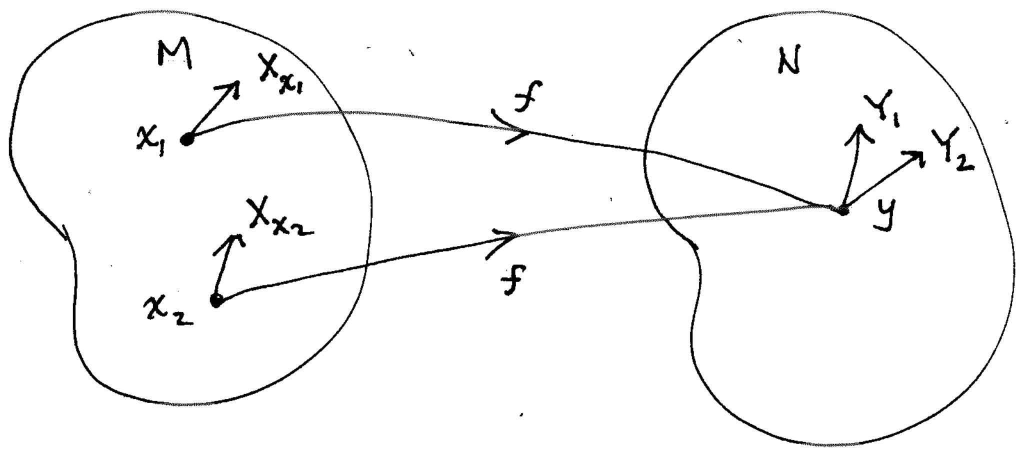
Suppose we have a map $f: M \to N$ between two manifolds, points $x \in M$, $y \in N$ such that $y = f(x)$. Also let $X \in T_x M$ and $Y \in T_y N$, with $Y = f_* X$. f_* is the derivative map, tangent map, or push-forward.



Here $f_* : T_x M \to T_y N = T_{f(x)} N$ is a linear map between tangent spaces.

Now suppose we have a vector field on M, $X \in \mathfrak{X}(M)$. We use the same symbol X for the vector field as we used a moment ago for a vector at a point. Can we push X forward to create a vector field on N?

In general, no. Suppose f is not injective. Then we can have 2 points $x_1, x_2 \in M$, $x_1 \neq x_2$, that map to the same point $y \in N$:



and the vector field X can be evaluated at points x_1 and x_2 giving individual vectors $X_{x_1} \in T_{x_1}M$ and $X_{x_2} \in T_{x_2}M$. These can be mapped by the two different versions of f_*,

$$f_*|_{x_1} : T_{x_1}M \to T_yN$$

$$f_*|_{x_2} : T_{x_2}M \to T_yN$$

but when these maps are applied to X_{x_1} and X_{x_2}, they give in general two different vectors $Y_1, Y_2 \in T_yN$. So this can't be a vector field on N because it has two values at one point.

Ok, so suppose f is injective, so the above does not happen. If f is not surjective, then there are points $y \in N$ that do not lie in $\operatorname{img} f$, so there is no vector Y pushed forward from X on M at such points. Again, we fail to get a vector field on N (although we do have one on $\operatorname{img} f$).

So, to get a vector field on N, f must be both injective and surjective, that is, it must be a <u>bijection</u>.

In that case it makes sense to talk about the push forward of $X \in \mathfrak{X}(M)$ to $Y \in \mathfrak{X}(N)$, for which we can write

$$Y = f_* X .$$

This is a different use of the notation f_* previously, a map between specific tangent spaces; now f_* is a map: $\mathfrak{X}(M) \to \mathfrak{X}(N)$, defined when f is a diffeomorphism.

Now, what about the pull-back of covectors (also called 1-forms). Can we apply this to fields?

Let $\beta \in \mathfrak{X}^*(N)$ be a covector field on N, let $y = f(x)$ as above. Then we can define $\alpha \in \mathfrak{X}^*(M)$ by using the pull back. At point $x \in M$, we define

$$\cancel{(\alpha|_x)(X) = (f^*(\beta|_y))(f_* X)}$$

$$\alpha|_x = f^* \beta|_y$$

by

$$(\alpha|_x)(X) = (\beta|_y)(f_* X), \qquad X \in T_x M.$$

This can be written

$$(f^*\beta)|_x (X) = \beta_{f(x)} (f_* X), \qquad X \in T_x M.$$

This is the same as the definition of the pull back discussed earlier, but now we are making the point that this formula can be applied at every point $x \in M$, regardless of the nature of f (injective or not; ~~surj~~ surjective or not; etc.).

To summarize, vector <u>fields</u> can be pushed forward by f_* only if f is a diffeomorphism, while covector <u>fields</u> can always be pulled back, regardless of the nature of f (of course we assume f is smooth).

Now XY is not a vector field (because it is a 2nd order operator), but it turns out that $XY-YX$ *is* a vector field (all 2nd derivs cancel), and, in fact,

$$\boxed{\mathcal{L}_X Y = [X,Y]}$$

This is called the Lie Bracket

The (important) commutator has the following properties:

a) $[X,Y] = -[Y,X]$

b) $[X,Y]$ linear in X, Y (over $\mathbb{R}$)

c) $[X,[Y,Z]] + \circlearrowright = 0$ (Jacobi).

The set $\mathfrak{X}(M)$ forms a Lie algebra. (of diffeomorphism group).

Some other properties of the commutator:

(a) $f_*[X,Y] = [f_*X, f_*Y]$ when $f: M \to N$ is a diffeomorphism

(b) $\mathcal{L}_{[X,Y]} = [\mathcal{L}_X, \mathcal{L}_Y]$

→ bec. advance maps commute w. diffeomorphisms.

(a) is almost obvious; a diffeomorphism is an isomorphism of differentiable structure, integral curves mapped into integral curves etc.

Now we turn to differential forms. A diff. form of rank r is a completely antisymmetric type $(0,r)$ tensor. Why antisymmetric? Because you need these for integrating over oriented, r-dimensional surfaces. Consider an example from 3D vector calculus. Let a small area element be specified by two small vectors $\vec{\xi}$ and $\vec{\eta}$. These might define a small element of a 2D surface.