Recall, a tensor of type (r,s) at a point $p \in M$ is a multilinear map,

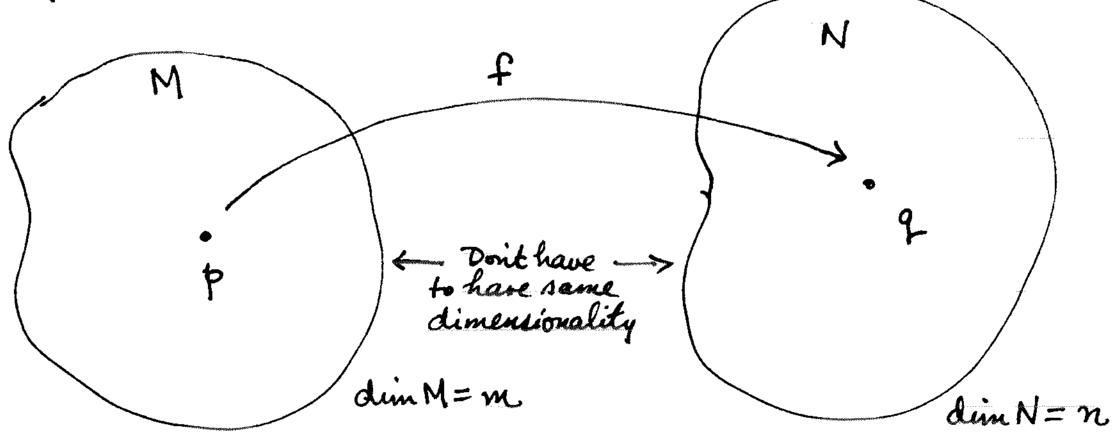
$$T: \underbrace{T_p^*M \times \ldots \times T_p^*M}_{r \text{ copies}} \times \underbrace{T_pM \times \ldots \times T_pM}_{s \text{ copies}} \to \mathbb{R}.$$

The components of T are given by (w.r.t. a chart)

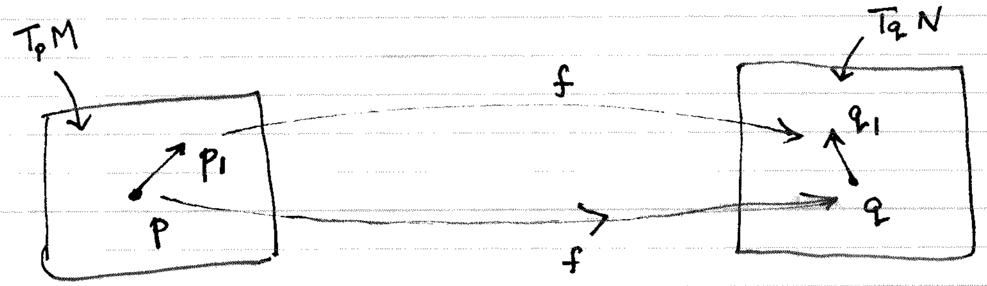
$$T^{i_1 \ldots i_r}{}_{j_1 \ldots j_s} = T\left(dx^{i_1}, \ldots, dx^{i_r}; \frac{\partial}{\partial x^{j_1}}, \ldots, \frac{\partial}{\partial x^{j_s}}\right).$$

A tensor field on M is an assignment of a tensor at each point $p \in M$. The components of a tensor field are functions of position.

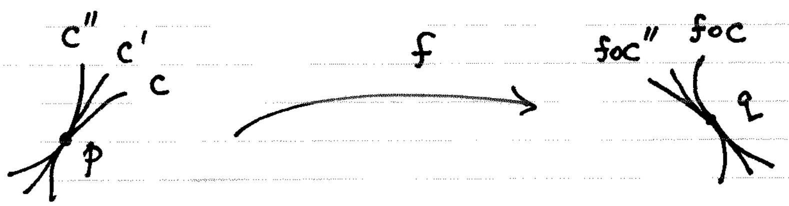
Now we consider the behavior of fields under maps. Let $f: M \to N$ be a map between manifolds, let $p \in M$ and $q \in N$ be points such that $q = f(p)$.



Question: Given $X \in T_pM$, is there any way to associate it with a $Y \in T_qN$? Yes, use the small displacement interpretation of a tangent vector (p_1 close to p), and define $q_1 = f(p_1)$. (You map both the base and the tip of the small arrow under f, to get a new small arrow on N. Both are understood to be displacements taking place in some elapsed parameter Δt.)



Alternatively, in the equivalence class of curves interpretation, just map the curves themselves:



Thus we can say, if $X = [c]$, then $Y = [f \circ c]$. This defines a linear map

$$f_* : T_pM \to T_qN$$

where f_* is called the <u>tangent map</u>, <u>derivative map</u>, or <u>push-forward</u>. Note that f_* can be defined succinctly by

$$f_*[c] = [f \circ c].$$

If we impose <u>coordinates</u> (charts) on M and N (containing p and q), we can write f_* in coordinates. Let x^i be coordinates on M and y^i those on M. Let

$$X = \sum_{i=1}^{m} X^i \left.\frac{\partial}{\partial x^i}\right|_p$$

$$f_*X = Y = \sum_{i=1}^{n} Y^i \left.\frac{\partial}{\partial y^i}\right|_q$$

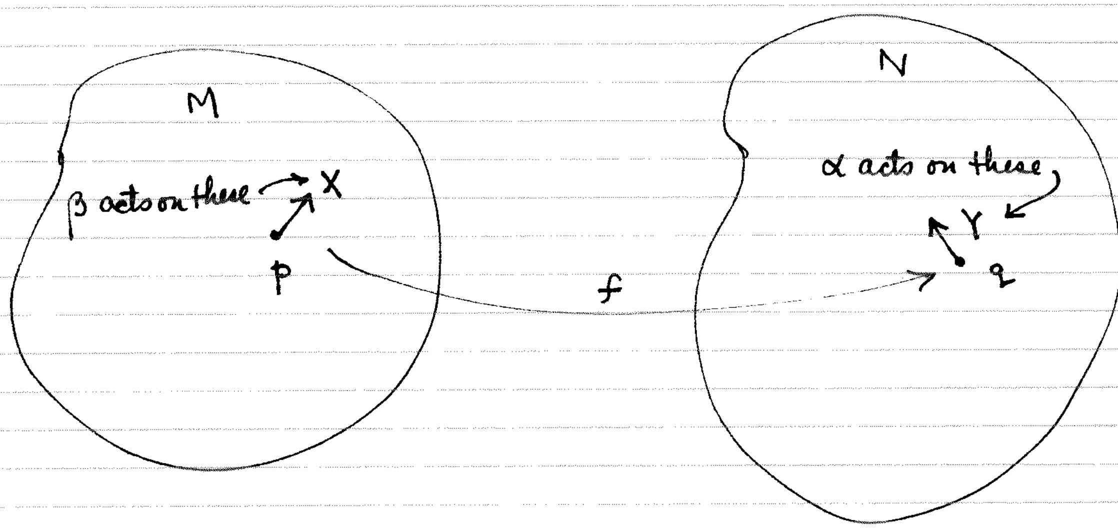
Then

this is just the chain rule, if you think $X^i = \frac{dx^i}{dt}$, $Y^i = \frac{dy^i}{dt}$.

$$Y^i = \sum_j \frac{\partial y^i}{\partial x^j} X^j ,$$

where $\partial y^i/\partial x^j$ is the coordinate representation of the derivatives of f.

This is how we map vectors. For covectors, it works the other way, i.e., given a covector $\alpha \in T_q^* N$, we can associate it with another covector $\beta \in T_p^* M$.



That is, we define a map $f^* : T_q^* N \to T_p^* M$ by demanding $\beta(X) = \alpha(Y)$ when $Y = f_* X$. That is, $\beta = f^*\alpha$ is defined by

$$(f^*\alpha)(X) = \alpha(f_* X), \qquad \forall\ X \in T_p M.$$

The covector $f^*\alpha \in T_p^* M$ is said to be the <u>pull-back</u> of $\alpha \in T_q^* N$, because it works in the opposite direction to f.

This is for a covector at a point (two of them, α and $f^*\alpha$). If now we let α be a covector field (same symbol, new meaning), $\alpha \in \mathcal{X}^*(N)$, then $f^*\alpha \in \mathcal{X}^* M$, is given by

$$(f^*\alpha)_p(X) = \alpha_{f(p)}(f_* X), \quad \forall \quad p \in M$$

where the subscript indicates the point at which the field is evaluated. Thus 1-forms on N get pulled back into 1-forms on M.

A simpler example of the pull-back is for scalar fields. Let $\phi \in \mathcal{F}(N)$, $\phi: N \to \mathbb{R}$ be a scalar field. Then we define the pull back $\psi = f^*\phi \in \mathcal{F}(M)$ by

$$\psi(p) = (f^*\phi)(p) = \phi(f(p)) = (\phi \circ f)(p),$$

that is, $f^*\phi = \phi \circ f$.

Return to the pull-back of covectors, and put it into coordinate language. Let x^i, y^i be coordinates on M and N as above, and let $\beta = f^*\alpha$, so that

$$\alpha = \sum_{i=1}^{n} \alpha_i \, dy^i|_q$$

$$\beta = \sum_{i=1}^{m} \beta_i \, dx^i|_p$$

Then
$$\beta_i = \sum_{j=1}^{n} \frac{\partial y^j}{\partial x^i} \alpha_j .$$

Notice that f^{-1} need not be defined in order to define f_* and f^*. In particular, M and N need not have the same dimensionality. But if f^{-1} does exist (then $\dim M = \dim N$), then we can "push forward" covectors from M to N (by f^{-1*}) and

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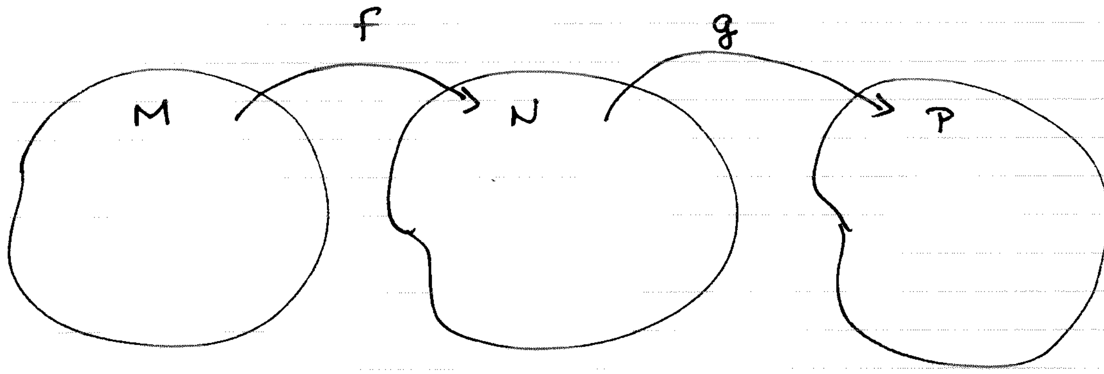
Behavior of f^*, f_* under compositions. Let $f: M \to N$, $g: N \to P$



Then

$$g \circ f : M \to P,$$

and $$(g \circ f)_* = g_* \circ f_* .$$

This is fairly obvious, you just map the small displacement vector (in M) under a succession of 2 linear maps, first by f_*, then g_*, to get $(g \circ f)_*$.

As for pull-backs, the rule is

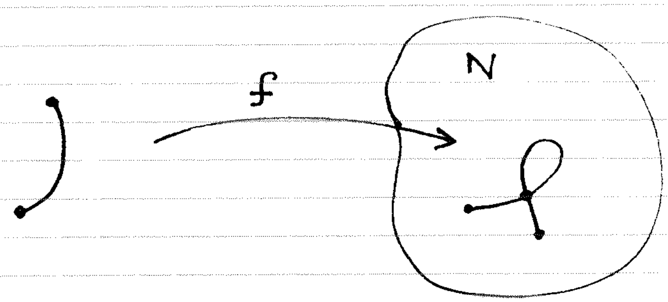
$$(g \circ f)^* = f^* \circ g^* \qquad \text{(in reverse order).}$$

"pull-back" vectors from N to M (by f_*^{-1}).

Now consider the mapping of one manifold of a certain dimensionality into one of higher dimensionality, $f: M \to N$, $\dim M \leq \dim N$. Then f is called an <u>immersion</u> if f_* is of maximal rank,

$$\text{rank } f_*: T_pM \to T_{f(p)}N = \dim M.$$

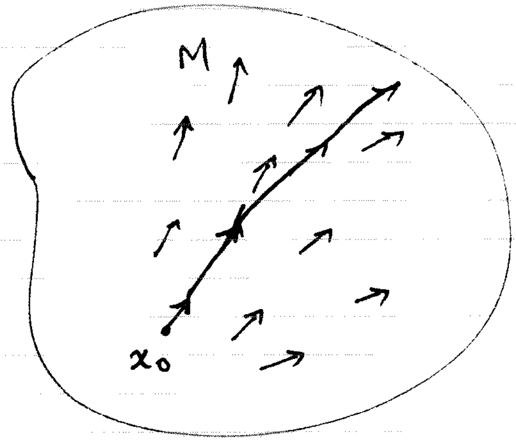
This means that each little piece of M, which looks like $\mathbb{R}^m$, gets mapped into a subset of N that also looks like $\mathbb{R}^m$. This means that f_* is injective (the image of M under f is locally $\overset{\dim M}{m}$-dimensional). However, an immersion does not preclude self-intersections:



To exclude self-intersections, we can demand that f itself be an injection. ~~This means~~ Then f is called an <u>imbedding</u> (because im f "looks like" M).

Now we consider <u>ordinary differential equations</u> (ODE's) and <u>flows</u>. Begin with an intuitive picture of a <u>vector field</u>, as a small displacement (each understood to be taking place in some elapsed parameter Δt), attached to each point of M:

– picture of $X \in \mathfrak{X}(M)$



Idea: Each point gets up and moves a small amount in time Δt.

If you just follow these arrows, starting with some initial point x_0, you trace out a curve called the integral curve of X. By following in integral curve for time t, starting at x_0, you get a final point x described by a function

at least sometimes it has this domain

$$\Phi : M \times \mathbb{R} \to M,$$

$$x = \Phi(x_0, t),$$

where Φ is called the advance map.

To make this more precise, express the vector field X in some chart:

$$X = \sum_i X^i(x) \frac{\partial}{\partial x^i}$$

x = a point
x^i = its coordinates
X = a vector
X^i = its components.

This is an operator which when acting on scalars f gives a number interpreted as df/dt. In particular, letting it act on the coordinates themselves gives a set of ODEs:

$$\frac{dx^i}{dt} = X^i(x).$$

Thus, a vector field on a manifold is a generalization of a system of ODE's on $\mathbb{R}^n$. Standard theorems on ODE's say that the system above has a unique solution $x^i(t)$ satisfying $x^i(0) = x^i_0$ for t in some interval

$[0,T]$, if the functions $X^i(x)$ are smooth. This is the (important) <u>uniqueness theorem</u> for ODE's (really, existence and uniqueness). However, even if the vector field $X^i(x)$ is smooth, the solution may not exist for all t (for example, it may run off to infinity in finite t). (For an example of this, consider $\dot{x} = x^2$, $x_0 = 1$ $(x \in \mathbb{R})$, for which $x \to \infty$ as $t \to 1$.) <u>For simplicity</u>, we will <u>assume</u> that this does not happen, i.e., that solutions $x^i(t)$ <u>exist for all time</u>, for any x_0^i. Then we can speak of the "general solution functions" $\Phi^i(t, x_0)$ that give $x^i(t)$, assuming $x^i(0) = x_0^i$. These ~~solution~~ functions satisfy, [we write $x^i(t) = \Phi^i(t, x_0)$, indicating explicitly the dependence on i.c.'s]

$$1) \quad \Phi^i(0, x_0) = x_0^i$$

$$2) \quad \frac{\partial \Phi^i}{\partial t}(t, x_0) = X^i(\Phi(t, x_0)).$$

These are just the initial conditions and ODE's expressed in terms of Φ^i. [We would normally write them,

$$1) \quad x^i(0) = x_0^i$$

$$2) \quad \frac{dx^i}{dt} = X^i(x(t)) .\quad].$$

All of the above is in one chart. By mapping a solution $x^i(t)$ in the given chart back onto M, we get a segment of an integral curve. Just before we run off one chart we can switch to another, thereby continuing the integral curve. The result is that we define a map $\Phi : \mathbb{R} \times M \to M$ or maps $\Phi_t : M \to M$ (a different notation), such that