Now we develop notation for groups, in order to talk about $\pi_1(M)$ for various M.

Let G be a group and $X = \{a, b, c, \dots\}$ a finite collection of elements of G. If every element $g \in G$ can be written as finite products of powers of $a, b, c, \dots$, written in some order, including negative powers, then G is said to be <u>finitely generated</u> and X to be ~~the~~ a set of <u>generators</u>. For example, we might have aba^{-1}, $ab^{-2}cb^{2}a^{3}$, etc. If two powers of identical generators are adjacent, they may be combined, e.g. $ab^2aa^{-2}c = ab^2a^{-1}c$. If any generator to the zero power occurs, we drop it, since $a^0 = e = 1 =$ identity. A product of generators obeying these rules will be called <u>reduced</u>. If every element of G can be written <u>uniquely</u> as a reduced product of generators, then G is said to be <u>freely generated</u>.

If G is finitely generated by $X \subset G$ but not freely, then there are <u>relations</u> connecting the generators. For example, an Abelian group with two generators $X = \{a, b\}$ will satisfy the relation $ab = ba$. This means that elements $g \in G$ can be written as reduced products in more than one way. We handle this case by going back to the free group, and then dividing by a subgroup.

Let F be the free group constructed out of n generators $(x_1, \dots, x_n)$. Think of the generators as an <u>alphabet</u>. A <u>word</u> is a finite product of powers of letters,

$$x_{j_1}^{i_1} x_{j_2}^{i_2} \cdots x_{j_s}^{i_s},$$

where $i_k \in \mathbb{Z}$ and $1 \le j_k \le n$. If a word is reduced according to the rules above, then it is said to be reduced. Let $F =$ set of reduced words. F acquires the structure of a group when we define multiplication by catenating, then reducing ~~words~~ the product. The identity is the word of length zero.

Suppose G is generated by $\{x_1, \dots, x_n\}$ but not freely. Then we can define a map

$$f : F \to G$$

which just maps reduced words in F into the same expression in G. However, this map is not injective if G is not free, i.e., there will be more than one reduced word in F that gives rise to a given element of G (for example, xy and yx if G is Abelian). The map is, however, surjective, since we are assuming the $\{x_1, \dots, x_n\}$ generate G. The map f is also a group homomorphism, because the rules for combining words in F ~~is the~~ also work in G. Therefore $G = F/\ker f$, where $\ker f$ is the normal subgroup of ~~G~~ F that is mapped onto the identity in G.

~~Words in ker f have the form grg^{-1}, where r is a relation~~

$\ker f$ is specified by constraints on the generators, also called relations. Some examples will give the idea.

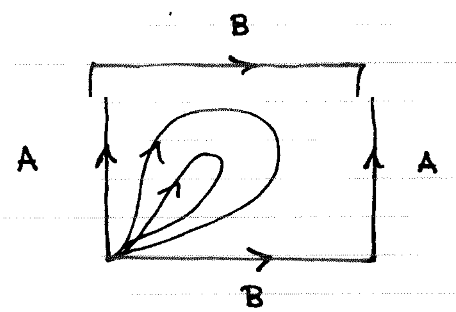
Let $G =$ Abelian ~~free~~ $\mathbb{Z}^2$ group, gen. by $\{x, y\}$. The relation is $xyx^{-1}y^{-1} = 1$. We write $G = \{x, y \,;\, xyx^{-1}y^{-1}\}$ as the presentation of G. Another example, $G = \mathbb{Z}_k$. Here we have one generator and one relation, $G = \{x \,;\, x^k\}$.

More formally, $C =$ constraint subgroup of ~~G~~ F is defined by

$$C = \text{gen}\,\{ grg^{-1} \mid g \in \cancel{G}\, F,\ r \in R \}$$

where $R \subset$ ~~G~~ F is the set of relations. C is a normal subgroup

Now some more homotopy groups. First the torus:

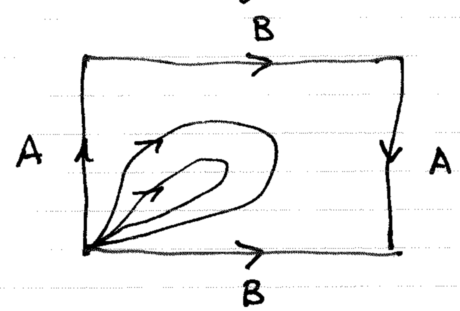


all 4 corners are same point.

Square with opposite sides identified. Obvious guesses for generators of $\pi_1(T^2)$ are loops A, B. But clearly $ABA^{-1}B^{-1}$ is contractible, so the group is

$$\pi_1(T^2) = \{A, B;\ ABA^{-1}B^{-1}\} = \mathbb{Z}^2.$$

Next Klein bottle. Square with a twisted identification.



again, all 4 corners are same pt.

This time $ABAB^{-1}$ is contractible, and

$$\pi_1(\text{Klein bottle}) = \{A, B;\ ABAB^{-1}\}. = \text{non-Abelian}.$$

Now remark on relation between homotopy and homology. The relation involves the <u>commutator</u> <u>subgroup</u> of an arbitrary group, which I now explain.

(not same C as on last page)

Let G be any group and let C be generated by all elements of the form,

$$xyx^{-1}y^{-1}, \quad x, y \in G.$$

called the <u>commutator</u> of 2 group elements.

C is called the commutator subgroup.

Thm: C is normal. Proof: Let $g \in G$, then

$$g\,xyx^{-1}y^{-1}g^{-1} = (gxg^{-1})(gyg^{-1})(gx^{-1}g^{-1})(gy^{-1}g^{-1})$$

$$= x'y'x'^{-1}y'^{-1} \qquad \text{where } x' = gxg^{-1},\ y' = gyg^{-1}.$$

Hence $gCg^{-1} = C$, $\forall g \in G$. Thus the quotient group G/C is defined.

Thm: G/C is Abelian. It is a kind of Abelianized version of G.

Proof: Let $[g_1]$, $[g_2]$ be two cosets of C. Then

$$g_1g_2\underbrace{(g_2^{-1}g_1^{-1}g_2g_1)}_{\in C} = g_2g_1,$$

So $[g_1g_2] = [g_2g_1] = [g_1][g_2] = [g_2][g_1]$, $\frac{G}{C}$ is Abelian.

Thm: Let K be a simplicial complex, let $G = \pi_1(K)$ and $C =$ commutator subgp. of G. Then

$$H_1(K, \mathbb{Z}) \cong \frac{\pi_1(K)}{C}.$$

Proof omitted.

Eg. with Klein bottle, $\pi_1(M) = \{x, y;\ xyxy^{-1}\}$.

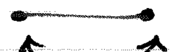
To divide by commutator subgroup, append extra relation $xyx^{-1}y^{-1}$.

If $xyxy^{-1} = 1$ and $xy = yx$, then $x(yx)y^{-1} = x(xy)y^{-1} = x^2 = 1$.

Thus, $H_1(\text{Klein})$ is the Abelian group generated by $\{x, y\}$ with the relation $x^2 = 1$, i.e., it is $\mathbb{Z} \times \mathbb{Z}_2$.

Now we examine the higher homotopy groups.

Recall that to study maps: $S^1 \to M$, we studied maps : $I \to M$ $\quad (I = [0,1] \subset \mathbb{R})$, subject to the condition that $f(0) = f(1)$ (these are loops). This was a matter of convenience.

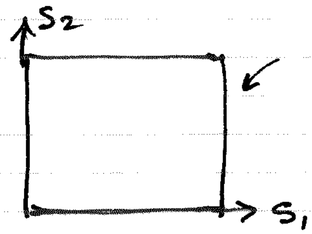
$I =$ 

bdry of I mapped to same pt.

Similarly, to study maps : $S^2 \to M$ it is convenient instead to look at maps

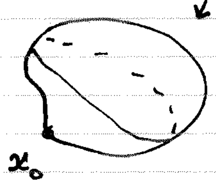
$$\alpha : I \times I \to M,$$

where $I \times I$ is the square and it is understood that the boundary of $I \times I$ is mapped to a single pt.

$I \times I$ 

whole bdy $\mapsto x_0 \in M$.

think im α



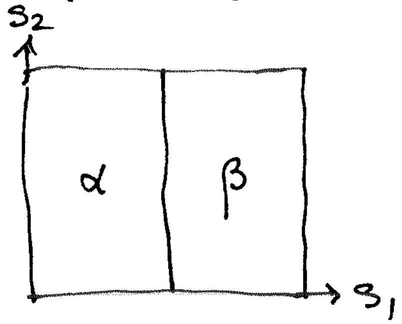
Note, square with all pts on bdry ∂I identified $\cong S^2$. Call such a map a 2-loop. Then the rest of the story proceeds very much as in the case of $\pi_1(M)$:

More generally, consider maps $\alpha : \underbrace{I \times \cdots \times I}_{I^n} \to M$, $\alpha : \partial I \to x_0$.
(These are n-loops.)

1. $\alpha \sim \beta$, 2-loops α and β are homotopic, if $\exists$ a smooth interpolating map (the homotopy) that preserves the boundary point. (i.e., maps the bdry to x_0 for all values of the deform. param.)
2. $\alpha \sim \beta$ is an equivalence relation, hence classes $[\alpha], [\beta]$ etc. meaningful.
3. $\alpha * \beta$ is defined by

$$(\alpha * \beta)(s_1, s_2) = \begin{cases} \alpha(2s_1, s_2) & 0 \le \tfrac{1}{2} \le s_1 \\ \beta(2s_1 - 1, s_2) & \tfrac{1}{2} \le s_1 \le 1. \end{cases}$$

Thus, pictorially:



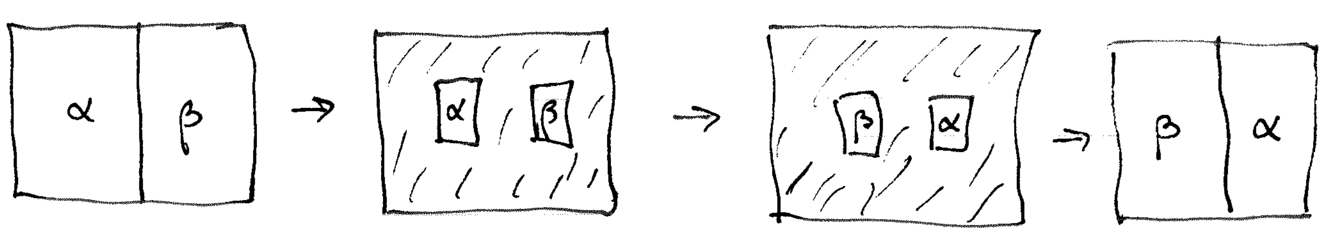
4. $[\alpha]*[\beta]$ is defined ($*$ respects homotopy classes), and other axioms of a group are satisfied. The group (for n-loops) is denoted $\pi_n(M, x_0)$.

5. $\pi_n(M, x_0)$ is isomorphic to $\pi_n(M, x_1)$, if M is arcwise connected. Hence we just write $\pi_n(M)$ for the abstract group (the n-th homotopy group).

6. If X is of same homotopy type as Y, then $\pi_n(X) = \pi_n(Y)$.

7. $\pi_n(X \times Y) = \pi_n(X) \times \pi_n(Y)$.

There is one important property of the higher homotopy groups ($n \geq 2$) not shared by π_1: The higher homotopy groups are Abelian. Reason for this can be seen pictorially:

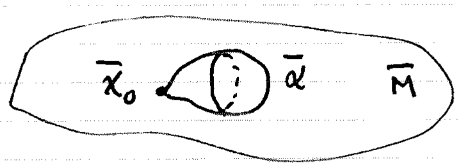


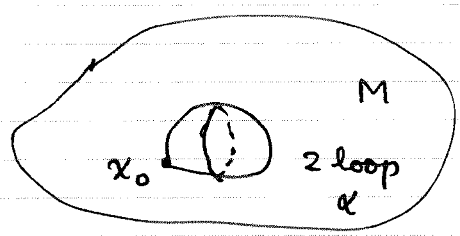
where the shaded region (and all boundaries) are mapped into x_0.

This shows that $\alpha * \beta \sim \beta * \alpha$, hence $[\alpha]*[\beta] = [\beta]*[\alpha]$.

Another property of higher homotopy groups not shared by π_1 concerns the universal covering space. Let M be a space and $\bar{M}$ its universal cover. Then $\pi_1(\bar{M}) = \{e\}$ ($\bar{M}$ is simply connected), but in general $\pi_1(M)$ is not trivial (if it is trivial, then $M = \bar{M}$).

But for $n \geq 2$, $\pi_n(\bar{M}) = \pi_n(M)$. The basic idea behind this fact is that the spheres S^n are simply connected for $n \geq 2$, so it's possible to define the lift of an n-loop:





The lift of a point x on the 2-loop α is the equivalence class $[(x,\gamma)]$, where γ is a path from x_0 to x confined to α. Since the 2-loop α is simply connected, it means that the class $[(x,\gamma)]$ is independent of the choice of γ, and therefore specifies a unique pt in $\bar{M}$.

As an application of this, note that $\mathbb{RP}^n$ is covered by S^n, so

$\pi_k(\mathbb{RP}^n) = \pi_k(S^n)$. So what is $\pi_k(S^n)$?

simplest case is $\pi_n(S^n)$, which concerns mappings of $S^n \to S^n$. Recall $\pi_1(S^1) = \mathbb{Z}$, where $n \in \mathbb{Z}$ is interpreted as a winding number. There is a generalization of this to higher dimensions, i.e., for a map $f: S^n \to S^n$ it is possible to say "how many times" the image of f "wraps around" S^n. This number is called the Brouwer ~~index of~~ degree of f. And, as in the case $n=1$, it turns out that the Brouwer degree uniquely characterizes the homotopy classes. Thus,

$$\pi_n(S^n) = \mathbb{Z} \qquad (\text{all } n \geq 1).$$

What about the case $\pi_k(S^n)$ for $k<n$? Recall $\pi_1(S^n) = \{e\}$ for $n \geq 2$ (the loop is contractible on the face of S^n). Something like this also happens for $\pi_k(S^n)$ when $k<n$, that is,

$$\pi_k(S^n) = \{e\}, \qquad 1 \leq k < n.$$

It turns out the case $k>n$ is also interesting. It's not easy to see this, because the highest dimensional case that is easy to visualize concerns maps $f: S^2 \to S^1$ $(k=2, n=1)$, and all such maps are trivial (= contractible). (Think of mapping a sphere S^2 onto the equator). Thus,

$$\pi_2(S^1) = \{e\}.$$

But it turns out there are nontrivial (not homotopic to constant map) maps from S^3 to S^2. The Hopf map discussed earlier in the course is an example of one of these. In fact,

$$\pi_3(S^2) = \mathbb{Z}.$$

In all these results, $k>1$, we can replace S^n by $\mathbb{RP}^n$.

See the table in Nakahara, p. 151.

Finally, what about the case $n=0$? A literal interpretation of the definition of $\pi_n(M, x_0)$ to the case $n=0$ would ~~be~~ involve maps $f: S^0 \to M$ with one point fixed. S^0 is properly just 2 points (± 1 in $\mathbb{R}$), so we can say $f(1) = x_0$, $f(-1)=$ another pt. x_1 say. But if M is connected, then x_1 can be continuously pulled to x_0, and $\pi_0(M) = \{e\}$, all connected M.

Now go back to defects in CM systems, now that we know about higher homotopy groups. Make a table of various order parameter spaces (OPS's):

System	OPS $= M$
Liqu. ^4He xy spin model	S^1
Nematic liquids	$\mathbb{RP}^2$
Dipole locked ^3He phase A	$SO(3) = \mathbb{RP}^3$

Recall, in 3D point defects are described by nontrivial homotopy classes $\pi_2(M)$, line defects (vortices) by $\pi_1(M)$, and textures (field configurations) with asymptotically constant values by $\pi_3(M)$.

Make a table of homotopy groups:

OPS	π_1	π_2	π_3
S^1	$\mathbb{Z}$	$\{e\}$	$\{e\}$
$\mathbb{R}P^2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}$
$SO(3) = \mathbb{R}P^3$	$\mathbb{Z}_2$	$\{e\}$	$\mathbb{Z}$
S^2	$\{e\}$	$\mathbb{Z}$	$\mathbb{Z}$

For example, a unit vector field $\hat{n}(\vec{r})$ (OPS $= S^2$) can possess point defects (characterized by an integer) and field configurations that are ~~with~~ asymptotically constant (also characterized by an integer), but no line defects (vortices).

Now start on manifold theory. Most of the spaces that occur in physics are differentiable manifolds. The idea is that a diff. manifold is a topological space with enough extra structure to talk about differentiability. That is, one can do calculus on manifolds.

Def: A differentiable manifold M is a topological space ~~such that~~ plus a set $\{U_i, \varphi_i\}$, where $\{U_i\}$ is an open cover of M (each U_i is an open subset of M, and $\underset{i}{\overset{\text{union}}{\bigcup}} U_i = M$.), and where φ_i is a map

$$\varphi_i : U_i \to \mathbb{R}^n$$

such that:

a) φ_i is a homeomorphism onto its image V_i in $\mathbb{R}^n$