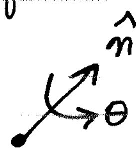
Take it as geometrically obvious that an arbitrary (proper) rotation can be written in axis-angle form:

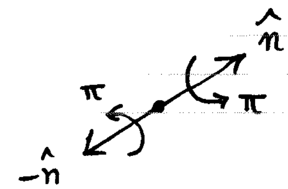
$\hat{n} \in S^2, \quad 0 \le \theta \le \pi$

$R(\hat{n},\theta) =$ 

using right hand rule.

The parameterization is unique except when $\theta = 0$, where $R(\hat{n},0) = I$ for any $\hat{n}$, and at $\theta = \pi$, where

$$R(\hat{n},\pi) = R(-\hat{n},\pi)$$



(same rotn)

So if we write $\vec{\theta} = \hat{n}\theta$, so that $\vec{\theta} \in \mathbb{R}^3$, then $SO(3)$ is identified with a sphere (the 3D, solid interior of a sphere in $\mathbb{R}^3$) out to a radius of π, including the surface (S^2) at $\theta = \pi$, but with antipodal points $(\hat{n},-\hat{n})$ on the surface identified. In other words,

$$SO(3) = \mathbb{R}P^3.$$

This is an example of a group manifold.

As for $SU(2)$, $\to$ physically, spin rotations. it is the set of 2×2, complex, unitary matrices with $\det = +1$:

$$U \in SU(2) \quad \Rightarrow \quad UU^+ = U^+U = I$$
$$\text{and } \det U = +1.$$

The condition $UU^+ = U^+U = I$ means that the rows and columns form pairs of orthonormal, complex, unit vectors (in $\mathbb{C}^2$). Write

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad a,b,c,d \in \mathbb{C},$$

Because of the conditions $U^\dagger U = 1$, $\det U = +1$, the 4 complex components of U satisfy certain constraints, and U can be written in terms of 4 real parameters $\underbrace{(x_0, x_1, x_2, x_3)}_{\equiv (x_0, \vec{x})} \in \mathbb{R}^4$

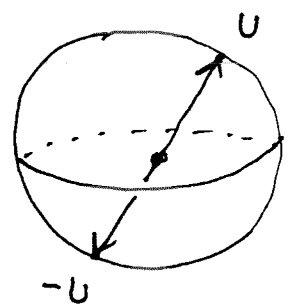
$$\underset{SU(2)}{\overset{\displaystyle U}{\cap}} = x_0 I - i\vec{x}\cdot\vec{\sigma} = \begin{pmatrix} x_0 - ix_3 & -x_2 - ix_1 \\ x_2 - ix_1 & x_0 + ix_3 \end{pmatrix}$$

where $x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1$. The (x_0, x_1, x_2, x_3) are the Cayley-Klein parameters, and they show that topologically,

$$SU(2) = S^3.$$

The relation between $SU(2) = S^3$ and $SO(3) = \mathbb{R}P^3$ is just the identification of antipodal points U and $-U$ in S^3 with a single element $R \in SO(3) = \mathbb{R}P^3$.

"equator" is S^2 →



← surface supposed to represent S^3 in $\mathbb{R}^4$

SO(3) can be thought of as the "northern hemisphere" with antipodal points on the "equator" (S^2) identified. This is the solid ball picture of SO(3) (in $\vec{\theta}$ coordinates).

Motivation for studying relationship between SO(3) and SU(2). Consider evolution of spin 1/2 particle (✓ neutron maybe) in magnetic field $\vec{B} = \vec{B}(t)$ which we allow to be time-dep. Define

$$\vec{\omega}(t) = g \frac{e}{2mc} \vec{B}(t)$$

a vector with dimensions of frequency (g = g-factor of particle). Let $\chi = \begin{pmatrix} \chi_\uparrow \\ \chi_\downarrow \end{pmatrix} \in \mathbb{C}^2$ be the usual spinor. The Schrödinger eqn is

$$i\hbar \frac{\partial \chi}{\partial t} = \vec{\omega}(t) \cdot \left(\frac{\hbar}{2}\vec{\sigma}\right) \chi \qquad \text{(Qu)}$$

where $\frac{\hbar}{2}\vec{\sigma}$ is the spin operator. Let $\vec{S}(t)$ be the expectation value of the spin operator,

$$\vec{S}(t) = \langle \chi(t) | \tfrac{\hbar}{2}\vec{\sigma} | \chi(t) \rangle$$

so that $\vec{S}$ is a c-number vector (not a vector of operators, $\vec{S} \in \mathbb{R}^3$). Then

$$\frac{d\vec{S}}{dt} = \vec{\omega}(t) \times \vec{S} \qquad \text{(Cl)}.$$

(Qu) is the "quantum eqn" and (Cl) is the "classical" eqn. (classical in the sense that eqns just like this occur in classical mechanics, they are the Euler equations). The solutions of (Qu) and (Cl) are

$$\chi(t) = U(t)\chi_0, \qquad U(t) \in SU(2)$$

$$\vec{S}(t) = R(t)\vec{S}_0, \qquad R(t) \in SO(3).$$

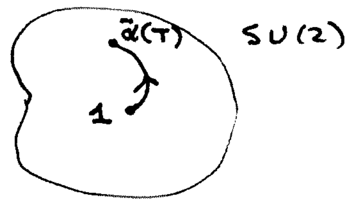
where $U(0) = 1$ (the 2x2 identity) and $R(t) = I$ (the 3x3 identity). The functions $U(t)$ and $R(t)$ are actually <u>paths</u> on the group manifolds SU(2) and SO(3). Let

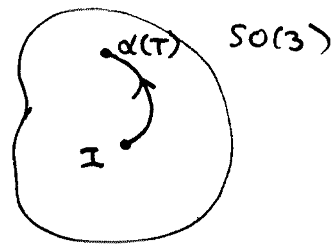
(T = final time)

$$\alpha : [0,T] \to SO(3)$$

$$\bar{\alpha} : [0,T] \to SU(2)$$

be two paths in $SO(3)$ and $SU(2)$, where $\alpha(t)$ means $R(t)$ and $\bar{\alpha}(t)$ means $U(t)$, satisfying $\bar{\alpha}(0) = 1$, $\alpha(0) = I$. Picture on the group manifolds,





Consider the stmt: "If you rotate a neutron by $360°$, it doesn't return to its original self but rather undergoes a phase change of -1. You have to rotate it by $720°$ to make it return to itself." Actually it is not the final value of the classical rotation $R(t)$ (or $\bar{\alpha}(t)$) that determines the outcome, but rather the history. Here is a correct stmt: Let $R(T) = \alpha(T) = I$ (at $t=T$). Then $\alpha : [0,T] \to SO(3)$ is a loop based at I. But $SO(3) = \mathbb{RP}^3$ (topologically speaking), so there are two homotopy classes the loop α can be in, the trivial class or the nontrivial class, since $\pi_1(\mathbb{RP}^3) = \mathbb{Z}_2$. Then

$$U(T) = \bar{\alpha}(T) = \begin{cases} +1 & \text{if } \alpha \in \text{trivial (contractible) class} \\ -1 & \text{if } \alpha \in \text{other class.} \end{cases}$$

The final state of the neutron depends on the homotopy class of the loop α in $SO(3)$. In fact one may say that the existence of spin is related to this nontrivial homotopy group $\pi_1(SO(3)) = \mathbb{Z}_2$.

There is an important map $p: SU(2) \to SO(3)$ that occurs in this theory. (p stands for "projection.") It is defined by ...

$$R_{ij} = \frac{1}{2} \mathrm{tr} \left(U^{+} \sigma_i U \sigma_j \right) \qquad \text{where } U \in SU(2),$$

i.e., it defines a function $R(U)$ or $R = p(U)$. One can show that

$$R(t) = p(U(t))$$

in the spin problem, i.e., p maps the path $\bar{\alpha}(t)$ in $SU(2)$ into $\alpha(t)$ in $SO(3)$. Note that $p(U) = p(-U)$, so the inverse $p^{-1}(R)$ of $R \in SO(3)$ consists of 2 points U and $-U$ (it turns out there are only these two). p is a two-to-one projection.

$SU(2)$ is ~~an example of~~ said to be a double cover of $SO(3)$. This is an example of a space M (here $SO(3)$) and its covering space $\bar{M}$ (here $SU(2)$). The projection p in the general case ~~has~~ is a map $p: \bar{M} \to M$ from the covering to the covered spaces. The path $\bar{\alpha}(t)$ defined above in $\bar{M} = SU(2)$ is called the *lift* of the path $\alpha(t) = R(t)$ in $M = SO(3)$. We mention all this (as yet) undefined terminology to give an example a preview of what will come.

Covering spaces don't have to be groups, but in this example they are, and there is extra structure because of that. For example, $p: SU(2) \to SO(3)$ is a group homomorphism, with kernel $\{1, -1\}$ (the image is all of $SO(3)$).

Digression on quaternions. Quaternions don't have anything to do with homotopy, but they are related to SU(2), so we'll say something about them now. Recall any $u \in SU(2)$ can be written

$$u = x_0 1 - i \vec{x} \cdot \vec{\sigma}$$

where (x_0, x_1, x_2, x_3) are the Cayley-Klein parameters and $\sum_{i=0}^{3} x_i^2 = 1$.

Thus every element of SU(2) corresponds to a unit vector in $\mathbb{R}^4$ and vice versa, and $SU(2) \cong S^3 \subset \mathbb{R}^4$.

We get the quaternions if we drop the constraint on the 4 x's and let them run all over $\mathbb{R}^4$. Thus if $q \in \mathbb{H}$ (the set of quaternions), then

$$q = x_0 1 - i \vec{x} \cdot \vec{\sigma}$$

and $\mathbb{H} \cong \mathbb{R}^4$, with a multiplication rule given by the expression above and the algebra of Pauli matrices. Sometimes this is written

$$q = x_0 1 + x_1 \hat{\imath} + x_2 \hat{\jmath} + x_3 \hat{k},$$

where $\hat{\imath}^2 = \hat{\jmath}^2 = \hat{k}^2 = -1$. Here in a sense we have a representation of the quaternions by means of 2×2 matrices.

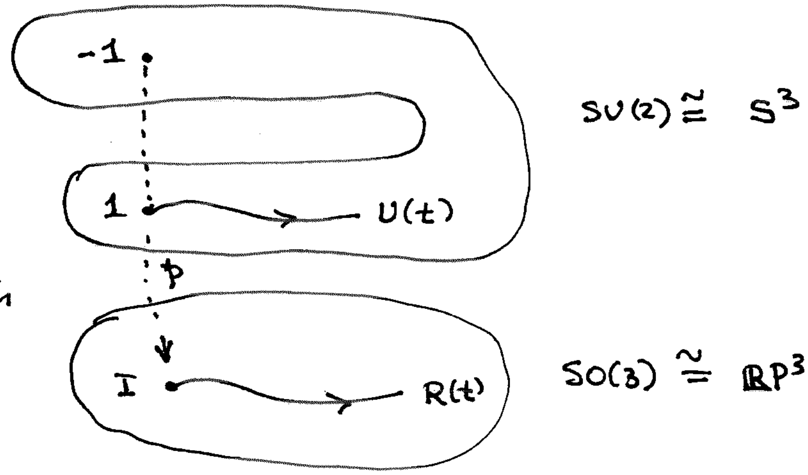
Obviously any vector in $\mathbb{R}^4$ can be written as a magnitude r times a unit vector, so

$$q = r u, \qquad \begin{matrix} 0 \le r \\ u \in SU(2) \end{matrix}$$

The elements of SU(2) are the unit quaternions. This is like the complex numbers, every $z \in \mathbb{C}$ can be written $z = re^{i\phi}$ where $0 \le r$ and $e^{i\phi} \in U(1)$. Multiplication of quaternions is noncommutative since SU(2) is non-abelian.

Return to SU(2) as a cover of SO(3). Let's divorce ourselves from the eqns. of evolution for $R(t)$ and $U(t)$ and look at topological aspects.

Not a very realistic representation of the relation betw. SU(2) and SO(3) →



Let $R(t)$ be a contin. path in SO(3) starting at $R(0) = I$. We wish to find a contin. path $U(t)$ in SU(2) such that $p: U(t) \mapsto R(t)$. Since p^{-1} is double-valued, there is no unique answer. But if we require that $U(0) = 1$ and that $U(t)$ be continuous, then the answer is unique. First, $U(0) = +1$ picks one of the two preimages at $t = 0$. Then, for each small step we make in $R(t)$, only one of the two possible preimages is possible if we demand $U(t)$ be continuous, because one of the preimages is close to the part of $U(t)$ developed already and the other is far away. Note that U and $-U$ are on opposite sides of S^3 and never come close together. The path $U(t)$ created in this way is the <u>lift</u> of $R(t)$. It is like the lifts we will see later in fiber bundle theory, except the law of "parallel transport" is determined by continuity.

Now let $R(t)$ be a loop, i.e. $R(0) = R(T) = I$. This loop belongs either to the trivial or nontrivial class of $\pi_1(SO(3)) = \mathbb{Z}_2$. We now show that $R(t)$ is trivial iff $U(T) = +1$, where $U(t)$ is the lift of $R(t)$. This tells us the state of the neutron after the classical rotation has returned to the identity.

If $R(t)$ is contractable (trivial), we wish to show that $U(T)=+1$. Suppose not, i.e. suppose $U(T)=-1$. Then the lift of the loop $R(t)$ is an open path $U(t)$ in $SU(2)$ that ends at $U(T)=-1$. Now as we contract $R(t)$ down to the constant loop $c: I \to SO(3): t \mapsto Id$, $\hookrightarrow [0,T]$ the lift of the contracting ~~curve~~ loop is a path $U(t)$ that must end on either $+1$ or -1, since $R(T)=I$. At the begining of the contraction, $U(T)=-1$ (we are supposing), while at the end ~~th~~ we have the lift of the constant loop in $SO(3)$ which is the constant loop : $[0,T] \to SU(2): t \mapsto 1$ in $SU(2)$. But the endpoint cannot change continuously from -1 to $+1$, while the lifting process is continuous. A continuous function that takes values in a discrete set must be constant. So the assumption $U(T)=-1$ must be wrong, and we must have $U(T)=+1$.

Conversely, suppose we have a loop $R(t)$ on $SO(3)$ that lifts into a loop $U(t)$ on $SU(2)$, ie a path that end in $U(T)=+1$. Since $SU(2) \cong S^3$ is simply connected, this loop is contractable. As it contracts, its projection onto $SO(3)$ contracts continuously to the constant loop. Therefore $R(t)$ belonged to the trivial class.

Now we give the official definition of a covering space.

Def. Let $M, \bar{M}$ be connected topological spaces with a map $p: \bar{M} \to M$ such that

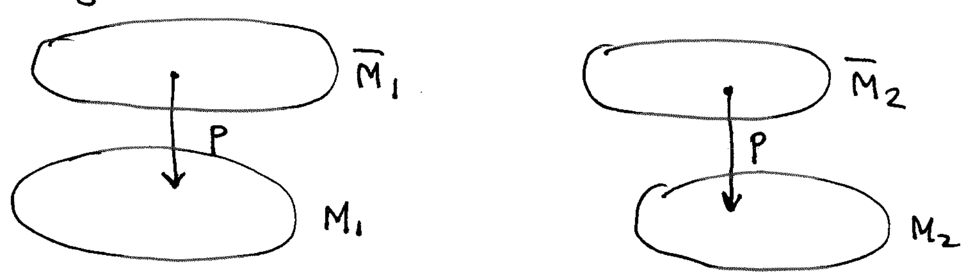
(M = "original" or "base" space; $\bar{M}$ = "covering" space)

(1) p is surjective

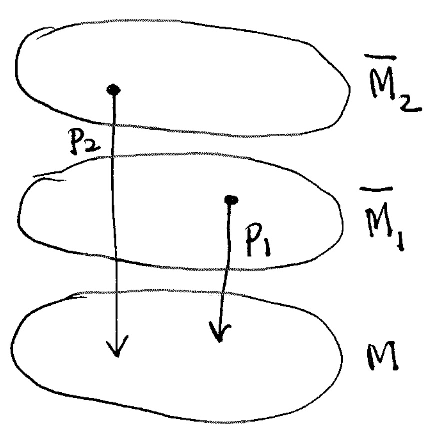
(2) for each $x \in M$ $\exists$ a connected open neighborhood $V \subset M$ containing x such that $p^{-1}(V)$ is a disjoint union of open sets $\{V_\alpha\}$ in $\bar{M}$, each mapped homeomorphically onto V by p, $p(V_\alpha) = V$, $\forall \alpha$.

Then $\bar{M}$ is ~~the~~ a <u>covering space</u> of M. If $\bar{M}$ is simply connected, then it is the <u>universal covering space</u>.

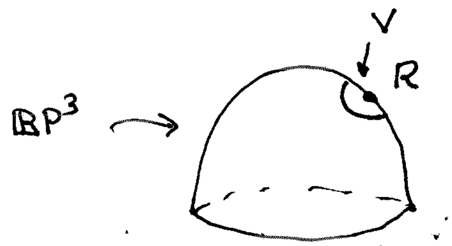
<u>Remarks.</u> We require M to be connected, because otherwise we might as well talk about the cover of each component (piece) of M separately:



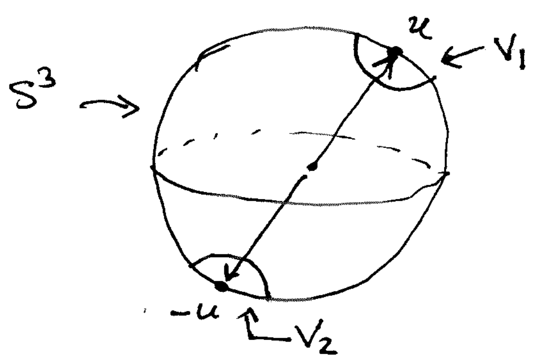
And we require $\bar{M}$ to be connected, because otherwise M can be thought of as having more than one cover, each of which can be treated separately:



Look at requirement (2), regarding open sets, in the case of $SU(2)$ covering $SO(3)$. Represent $SO(3)$ as the northern hemisphere of S^3, let $R \in SO(3)$, and let V be a neighborhood of it:



Then look at $p^{-1}(V)$ on $SU(2)$, represented as the whole of S^3:



$$p^{-1}(V) = V_1 \cup V_2$$

$$V_1 \cap V_2 = \phi$$

It means that in the neighborhood of each pre-image of $p^{-1}(R)$, $SU(2)$ "looks like" $SO(3)$, ~~topologically~~ speaking. The fact that $V_1 \cap V_2 = \phi$ means that the preimages $(u,-u)$ are well separated from each other, and remain so as R moves around on $SO(3)$. This is what allows us to make a unique choice of preimage, when we are lifting a curve and following it around by demanding continuity.

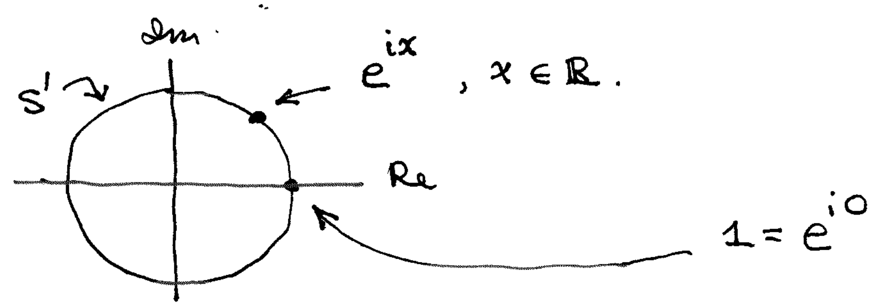
Apropos lifts, a theorem.

Thm. Let $\alpha : I \to M$ be a continuous path on M with $\alpha(0) = x_0$, and let $\bar{x}_0$ be a choice of a point in $p^{-1}(x_0)$. Then there exists a unique continuous path $\bar{\alpha} : I \to \bar{M}$ such that

$\bar{\alpha}(0) = \bar{x}_0$ and $p(\bar{\alpha}(t)) = \alpha(t)$, $t \in I$.

The covering space is intuitively an "unrolling" of the original space, e.g. $SU(2)$ is obtained by "unrolling" $SO(3)$ once. The metaphor is especially clear in the case of the circle. We now examine $\pi_1(S^1)$. Previously we argued on intuitive grounds that $\pi_1(S^1) = \mathbb{Z}$, wrapping rubber bands around doorknobs, etc. Now we outline a more rigorous proof. Nakahara's discussion of this point is hard to follow.

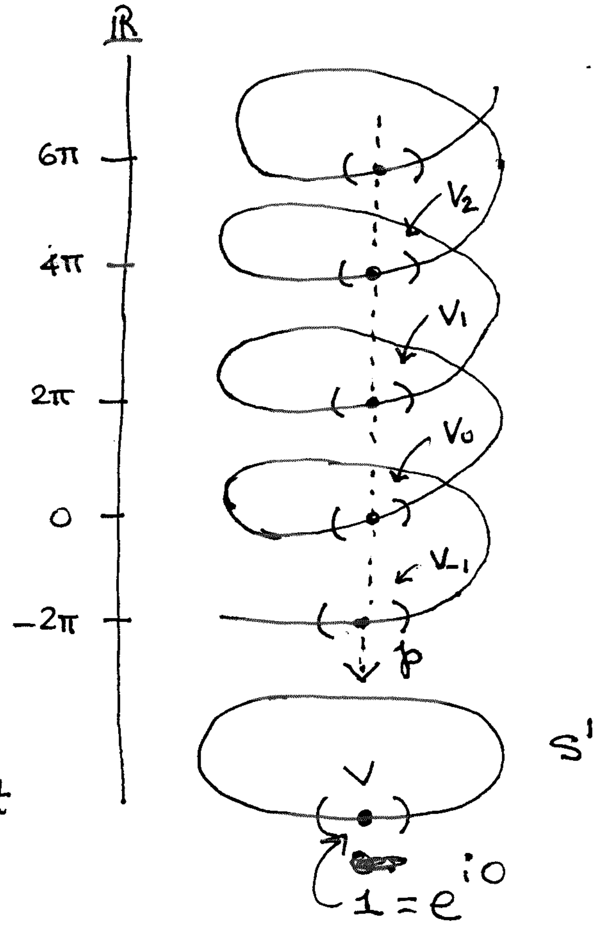
Identify S^1 with the unit circle in the complex plane:



Introduce map $p: \mathbb{R} \to S^1 : x \mapsto e^{ix}$. p^{-1} has an infinite number of branches, e.g. $p^{-1}(1) = \{\ldots, -2\pi, 0, 2\pi, 4\pi, \ldots\}$. $\mathbb{R}$ is a covering space of S^1. Wrap $\mathbb{R}$ around in a helix that sits over S^1 to make the projection more geometrical. Notice that $\exists$ an open interval V around $1 \in S^1$ s.t. $p^{-1}(V)$ is the union of an ∞ number of preimages V_α, disjoint. Each local part of the circle "looks like" the local part of $\mathbb{R}$

in a neighborhood of each preimage.

Thus, $\mathbb{R}$ is a covering space of S^1, in fact it is the universal cover since $\pi_1(\mathbb{R}) = \{e\}$ (since $\mathbb{R}$ is contractable).



Now let $\alpha : I \to S^1$ be a loop on S^1 based at 1, so

$$\alpha(0) = \alpha(1) = 1.$$

Let $\bar{\alpha} : I \to \mathbb{R}$ be the lifted loop starting at $\bar{\alpha}(0) = 0$ in $\mathbb{R}$. Since $\alpha(1) = 1$, $\bar{\alpha}(1)$ must belong to $p^{-1}(1)$, i.e. $\bar{\alpha}(1)$ must $= 2n\pi$ for some $n \in \mathbb{Z}$. Note that $\bar{\alpha}$ is not a loop if $n \neq 0$.

Now let α, α' be two loops on S^1 constructed as above with lifts $\bar{\alpha}, \bar{\alpha}'$ both starting at 0 and ending at $2\pi n$ and $2\pi n'$. Then we have that $\alpha \sim \alpha'$ iff $n = n'$.

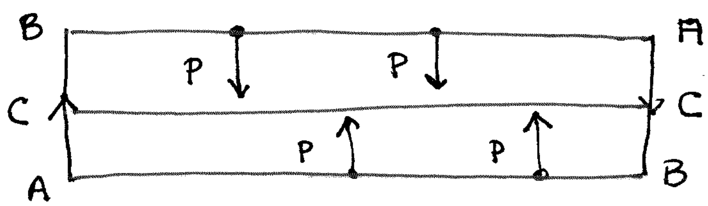
Proof. First suppose $\alpha \sim \alpha'$. Then there is a homotopy that deforms α into α'. Lift each curve of the deformation; this will create a homotopy deforming $\bar{\alpha}$ into $\bar{\alpha}'$. But the endpoint of $\bar{\alpha}$ or $\bar{\alpha}'$ cannot change discontinuously, and so cannot change at all. Thus $n = n'$. Conversely, suppose $n = n'$, so the lifted curves $\bar{\alpha}, \bar{\alpha}'$ start and end at the same points. Since $\mathbb{R}$ is simply connected, any two curves connecting the same points can be continuously deformed into one another. Do this, and use p to project the deformation onto S^1, whereupon we obtain a deformation of α into α'. Thus $\alpha \sim \alpha'$.

Thus we conclude that there is a one-to-one, onto mapping between homotopy classes in $\pi_1(S^1)$ and the integers $\mathbb{Z}$. With a little more effort, we can show that multiplication of homotopy classes corresponds to addition in $\mathbb{Z}$. Thus,

$$\pi_1(S^1) = \mathbb{Z}.$$

$\mathbb{R}$ is the <u>universal</u> <u>covering</u> <u>space</u> of S^1; S^1 has been "unrolled" an ∞ number of times.

It is not necessary to unroll all the way. Consider the single edge ABA of the Möbius strip, itself a circle, and the center line CC, which is also a circle.



Define p: ABA $\to$ CC, and ABA becomes a covering space of CC. One circle covers another (twice).

In another example, the 2-torus T^2 is a double cover of the Klein bottle. Finding the map $p: T^2 \to$ Klein will be left as an exercise. The 2-torus itself has a universal cover, $p: \mathbb{R}^2 \to T^2$. The plane is divided into parallelograms, with opposite sides identified.

It turns out that every connected space M has a covering space, although if M is simply connected then its only cover is M itself (covering once). If M is not simply connected, then it possesses one or more covers. The following is a construction of $\tilde{M}$, the universal covering space of M.

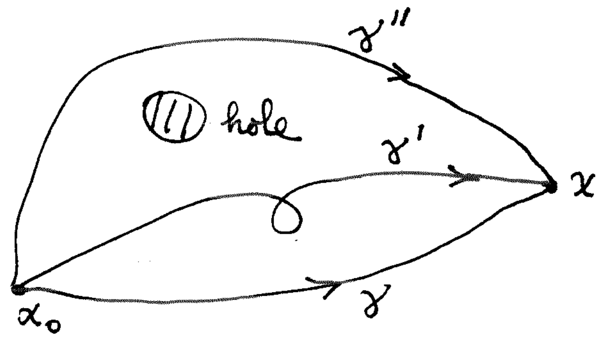
Let M be connected and $x_0 \in M$. Let $G = \pi_1(M, x_0)$, so elements $g \in G$ are equivalence classes of loops based at x_0.

Let (x, γ) be a (point, path) pair, where $x \in M$ and γ is a continuous path $: [0,1] \to M$, with $\gamma(0) = x_0$, $\gamma(1) = x$. It is redundant to write (x, γ), since $x = \gamma(1)$, but it is notationally convenient. Then let $(x, \gamma) \sim (x', \gamma')$ if $x = x'$ and γ homotopic to γ'. This is an equivalence relation. Define $\tilde{M}$ as the space of all equivalence classes,

$$\tilde{M} = \{ [(x, \gamma)] \mid x \in M, \ \gamma \text{ a path } x_0 \to x \}.$$

also, define $p: \bar{M} \to M : [(x,\gamma)] \mapsto x$.

(M)



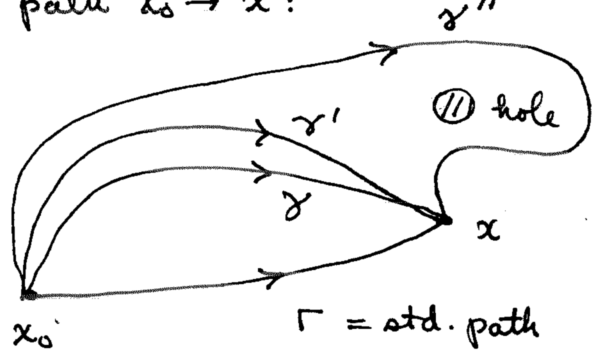
$[(x,\gamma)] = [(x,\gamma')]$ = one point of $\bar{M}$

$[(x,\gamma'')]$ = a different point of $\bar{M}$

We want to show that $\bar{M}$ is the universal cover of M, but first it is better to get some feel for what $\bar{M}$ looks like. To specify a point of $\bar{M}$ we must first choose a point $x \in M$ and then an equivalence class of curves connecting x_0 to x. The projection p just throws away the information regarding the equiv. class and returns x.

It turns out the equivalence classes of paths connecting x_0 to x can be labelled by elements of $G = \pi_1(M, x_0)$. This is obvious in the case $x = x_0$, since then the paths are loops based at x_0 and the equiv. classes of paths connecting x_0 to x <u>are</u> the elements of G. But it is true when $x \neq x_0$, too. Since G is a discrete group, the specification of a point $\bar{x} \in \bar{M}$ requires a point $x \in M$ plus a choice from a discrete set (i.e. G).

To label the homotopic equivalence classes of paths γ taking $x_0 \to x$, let Γ be a <u>standard</u> (fixed) path from x_0 to x and let γ be any path $x_0 \to x$:



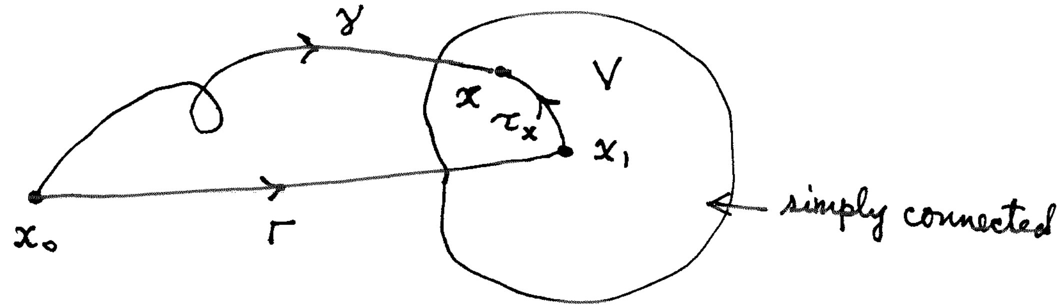
Then associate paths γ $(x_0 \to x)$ with loops α (based at x_0) by

$$\alpha = \gamma * \Gamma^{-1}.$$

Then it's easy to see that $\gamma \sim \gamma'$ (homotopic, as in picture above, but γ not $\sim \gamma''$) iff $\alpha \sim \alpha'$. Thus, equivalence classes of paths γ $(x_0 \to x)$ can be labelled by equivalence classes of loops α (based at x_0), i.e, by elements of $G = \pi_1(M, x_0)$. Therefore the number of preimages of $p^{-1}(x)$, for any $x \in M$, is the number of elements in G (the order of G). (This may be infinite.) The number of preimages of ~~p~~ x under p is independent of $x \in M$; it is "the number of times" $\bar{M}$ covers M.

But note that the labelling of points of $p^{-1}(x)$ by elements of G depends on the choice of the standard path Γ. Thus there arises the question of whether this labelling is the "same" as we had for points of $p^{-1}(x_0)$, that is, is it continuously connected with the labelling we used at x_0?

Here is a partial answer. It turns out we can label points of $p^{-1}(x)$ by elements of G in a continuous manner over any simply connected region of ~~E~~ M. Let V be such a region (x_0 need not be in V), let x_1 be a chosen point in V, and



let Γ be a standard path going from x_0 to x_1.

Now let x be any point in V and let τ_x be a path (inside V) from x_1 to x. We wish to use this to label points of $p^{-1}(x)$, i.e. equiv. classes of curves γ $(x_0 \to x)$. Again associate a path γ $(x_0 \to x)$ with a loop α based at x_0 by

$$\alpha = \gamma * \tau_x^{-1} * \Gamma^{-1}.$$

Again, $\alpha \sim \alpha'$ iff $\gamma \sim \gamma'$. This follows since Γ is fixed, and since V is simply connected, any 2 paths τ_x, τ'_x $(x_1 \to x)$ are homotopic. Thus, not only are points of $p^{-1}(x)$ labelled by elements of G (as before), but the labelling is continuous as x moves around inside V. This shows that

$$p^{-1}(V) \cong V \times G,$$

which, since G is a discrete set, means that $p^{-1}(V)$ consists of a discrete set of sets V_g $(g \in G)$ which are disjoint and homeomorphic to V. So $\bar{M}$ is a covering space, by the definition.

However, the continuous labelling of points of $p^{-1}(x)$ by elements $g \in G$ cannot be extended to all of M (unless it is simply connected, in which case G has only one element and $\bar{M} \cong M$.) So, in general, $\bar{M} \ncong M \times G$. In fiber bundle language, we would say that the fiber bundle is nontrivial.

Here are some exercises if you want to understand this construction better.

1. Show that the branches (points) of $p^{-1}(x)$ can be labelled continuously by elements of G as x moves along a path ~~mov~~ starting at x_0, as long as the path does not cross itself. What happens if the path does cross itself (what happens to the labelling)? In particular, suppose the path returns to x_0, making a loop?

2. Show that $\bar{M}$ is simply connected, hence the universal covering space.

One more remark. If M is a group manifold, then $\bar{M}$ (the universal covering group) can be given the structure of a group, and p is a homomorphism. This is the relation between $SU(2)$ and $SO(3)$, and $\mathbb{R}$ and S^1.