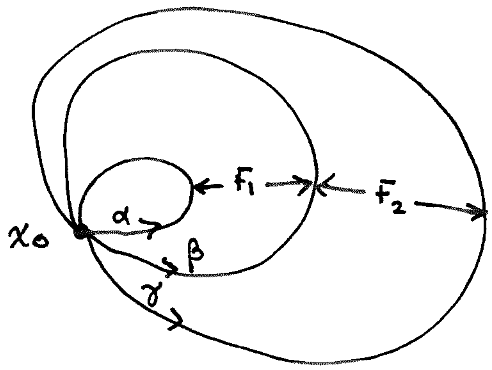
(a) $\alpha \sim \alpha$

(b) $\alpha \sim \beta \Rightarrow \beta \sim \alpha$

(c) $\alpha \sim \beta$ and $\beta \sim \gamma \Rightarrow \alpha \sim \gamma$.

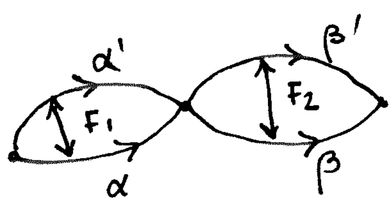
I will just prove (c), which is the hardest. A picture gives the idea:



Let $F(s,t) = \begin{cases} F_1(s,2t), & 0 \le t \le 1/2 \\ F_2(s,2t-1), & \frac{1}{2} \le t \le 1. \end{cases}$

Then F deforms α into γ.

Thus we have equivalence classes of loops based at a point x_0, $[\alpha]$, $[\beta]$, etc.
We want to define $[\alpha] * [\beta] = [\alpha * \beta]$ (multiplication of classes).
But we have to show that this is consistent (independent of choice made for representative element). That is, let $\alpha' \sim \alpha$ and $\beta' \sim \beta$. Want to show that $\alpha' * \beta' \sim \alpha * \beta$. Draw a picture, makes it obvious:



[Here picture drawn for arbitrary paths, but you get loops if endpoints are same point.]. Picture shows what to do:

Let $$F(s,t) = \begin{Bmatrix} F_1(2s,t), & 0 \le s \le 1/2 \\ F_2(2s-1,t), & \frac{1}{2} \le s \le 1 \end{Bmatrix}.$$

F is the homotopy that deforms $\alpha' * \beta'$ into $\alpha * \beta$.

Turns out, this $*$ law on equivalence classes of loops based at a point x_0 defines a group. Need to show:

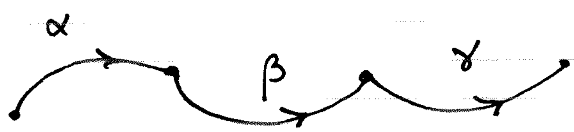
$$([\alpha][\beta])[\gamma] = [\alpha]([\beta][\gamma])$$ associative law (drop $*$ now)

$$[\alpha][c] = [c][\alpha] = [\alpha],$$ $[c]$ = equiv. class of constant ~~map~~ loop,

$$[\alpha^{-1}][\alpha] = [\alpha][\alpha^{-1}] = [c].$$ $c: [0,1] \to X: s \mapsto x_0$.

Thus $[c]$ is the identity: It is the equivalence class of loops that can be contracted to a point.

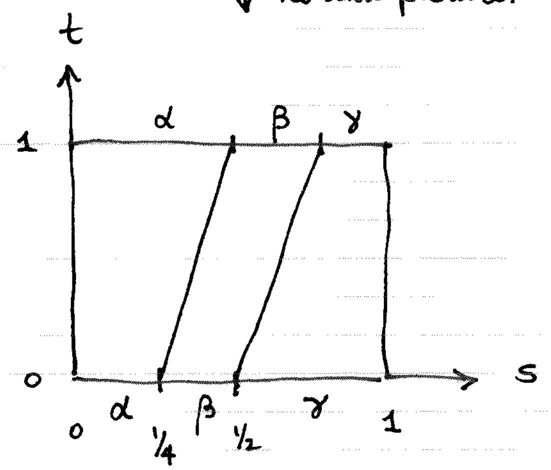
I will just prove the associative law (the others are easy). A picture makes it obvious, and also shows how to construct the formal proof:



$$(\alpha*\beta)(s) = \begin{cases} \alpha(2s), & 0 \le s \le 1/2 \\ \beta(2s-1), & 1/2 \le s \le 1 \end{cases}$$

deform according to this picture.

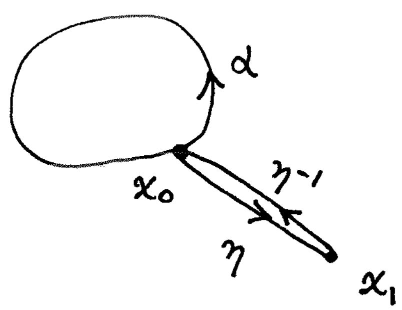
$$((\alpha*\beta)*\gamma)(s) = \begin{cases} \alpha(4s), & 0 \le s \le 1/4 \\ \beta(4s-1), & 1/4 \le s \le 1/2 \\ \gamma(2s-1), & \frac{1}{2} \le s \le 1. \end{cases}$$



Thus, the set of equivalence classes of loops based at a point x_0 constitutes a group, denoted

$$\pi_1(X, x_0) \quad = \text{first homotopy group of } X \text{ based at } x_0.$$

This group is defined relative to a base point x_0. If we choose a different base point we get a different group, say, $\pi_1(X, x_1)$. What is the relation between these? Answer is easy in an arc-wise connected space (which for us is just a connected space), in which a path $\eta : [0,1] \to X$, $\eta(0) = x_0$, $\eta(1) = x_1$ always exists.



Given α based at x_0, can create an α' based at x_1 by writing,

$$\alpha' = \eta^{-1} * \alpha * \eta.$$

Moreover, if $\alpha \sim \beta$, then $\alpha' \sim \beta'$, so the conjugation by η preserves the equivalence class structure. (Easy to show.) So we have a mapping parameterized by η,

$$P_\eta : \pi_1(X, x_0) \to \pi_1(X, x_1) : [\alpha] \mapsto [\eta^{-1} * \alpha * \eta]$$

In fact, P_η is a group isomorphism. We prove this by showing first that P_η is a homomorphism, then that P_η^{-1} is a homomorphism.

Proof that P_η is a homomorphism is easy.

Need to show that $(P_\eta[\alpha])(P_\eta[\beta]) = P_\eta[\alpha\beta]$ (omit $*$)

$$LHS = [\eta^{-1}\alpha\eta][\eta^{-1}\beta\eta] = [\eta^{-1}\alpha\eta\eta^{-1}\beta\eta] = [\eta^{-1}\alpha\beta\eta] = RHS.$$

Similary, consider $P_\eta^{-1} : \pi_1(X, x_1) \to \pi_1(X, x_0)$ defined by

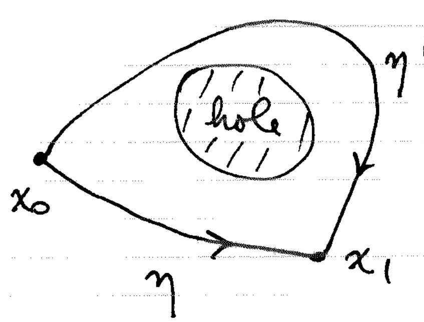
$$P_\eta^{-1}[\gamma] = [\eta\gamma\eta^{-1}]$$ where $\gamma \in \pi_1(X, x_1)$ and η as before.

Then easily show that P_η^{-1} actually is the inverse of P_η. Thus, P_η is a bijection, hence an isomorphism.

Thus, while $\pi_1(X, x_0)$ depends on the base point x_0, the group $\pi_1(X, x_1)$ at any other point (remember this is an arcwise-connected space) is isomorphic to it. Thus, as <u>abstract groups</u>, these groups are the <u>same</u>: it is written simply as

$\pi_1(X)$, called the <u>fundamental group of X</u>.

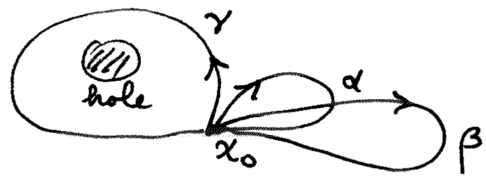
But notice, the isomorphism between $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ is not a <u>natural</u> isomorphism, because it depends on the choice of η connecting x_0 and x_1.



$P_\eta \neq P_{\eta'}$ in this picture.

Review: We are thinking of maps $f: S^1 \to M$ of the circle onto a given space M, in order to understand the topology of M and for other reasons. We usually represent these maps in a different form, $f: I \to M$ where $I = [0,1]$ is the unit interval and where we require $f(0) = f(1) \equiv x_0 \in M$. We call such a map a loop based at x_0.

Loops are considered equivalent (homotopic) if they can be continuously deformed into one another. The map that does the deformation is called the homotopy.



In the picture, $\alpha \sim \beta$ but α not $\sim \gamma$ (all loops based at x_0).
The set of equivalence classes of loops based at a point x_0 can be given a group structure, in which the multiplication is just catenation. The group is denoted $\pi_1(M, x_0)$, the first homotopy group based at x_0. The group $\pi_1(M, x_1)$ based at a different point is isomorphic to $\pi_1(M, x_0)$, if M is connected. Thus, as abstract groups, they are the same. This group is denoted $\pi_1(M)$, the fundamental group.

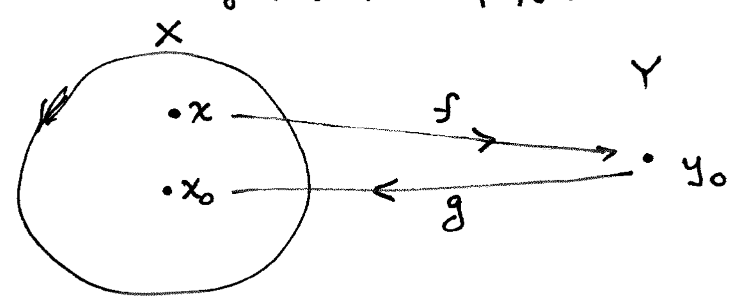
Now we explore some of the properties of the <u>fundamental group</u> $\pi_1(M)$ of a manifold M, and work on ways of computing it.

Recall defin of <u>homeomorphism</u>: Let X, Y be topological spaces. Then ~~X,Y~~ X, Y are homeomorphic ~~are homeomorphism~~ if $\exists$ contin maps $f: X \to Y$, $g: Y \to X$ such that $f \circ g = id_Y$ and $g \circ f = id_X$.

Now we introduce a weaker concept. Let X, Y be topological spaces.

<u>Def.</u> X and Y are of the <u>same homotopy type</u> if $\exists$ contin. maps $f: X \to Y$ and $g: Y \to X$ such that $f \circ g \sim id_Y$ and $g \circ f \sim id_X$. ($\sim$ means, "is homotopic to"). Point is that $=$ for a homeomorphism is replaced by $\sim$ for same homotopy type.

An <u>example</u> of spaces that are of same homotopy type but not homeomorphic. Let $X =$ 2-disk D^2, $Y =$ single point y_0 Let $f: X \to Y$ be the const. map $f(x) = y_0$, $\forall x \in X$. Let $g: Y \to X$ map y_0 onto a certain point $x_0 \in X$.
↑ center of disk.



Not homeomorphic, because f not invertible. ~~Well~~ Are X, Y of same homotopy type? Well, $f \circ g : Y \to Y : y_0 \mapsto y_0$ is just the identity map id_Y (has to be, since Y only has one point.). Next $g \circ f : X \to X$: x (any x) $\mapsto x_0$, it is the constant map. So is $g \circ f \sim id_X$? Yes, just shrink disk by radial factor, to make disk $\to$ central point.

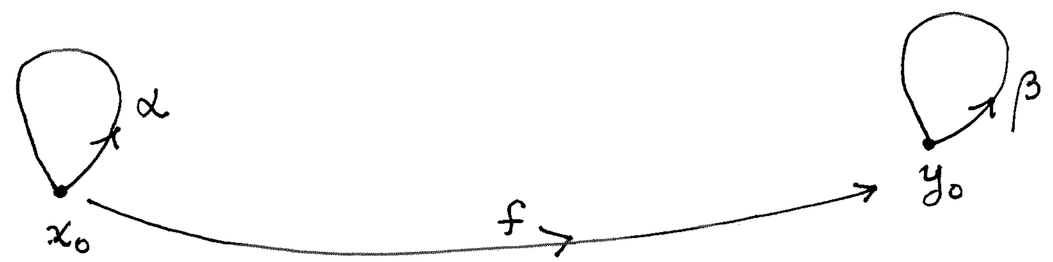
Notice that with "same homotopy type" (as with "homeomorphic")

Return to the theorem: If X, Y are of the same homotopy type, then

$$\pi_1(X, x_0) = \pi_1(Y, y_0)$$

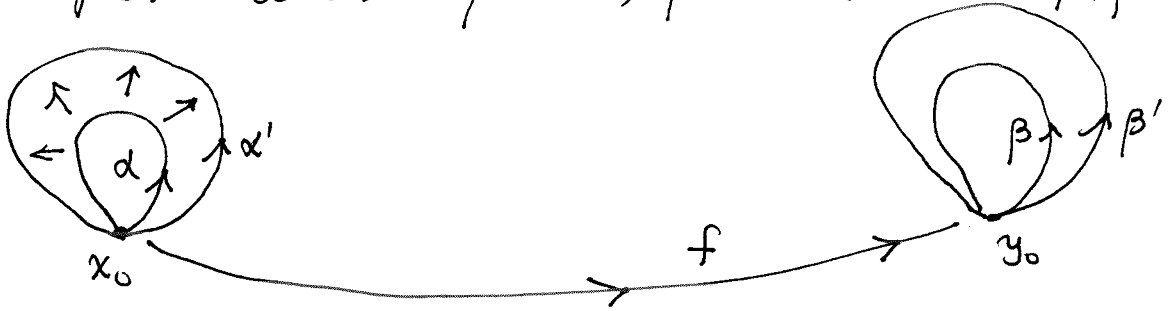
where $y_0 = f(x_0)$ and f is the map: $X \to Y$ that enters into the definition of "same homotopy type". Here "=" means, "is isomorphic to." Thus, if X, Y are connected, so that the homotopy groups are independent of base point, then $\pi_1(X) = \pi_1(Y)$.

Here is the beginning of a proof of the theorem. Choose $x_0 \in X$, let $y_0 = f(x_0) \in Y$, and let $\alpha : I \to X$ be a loop at x_0. Let β be the image of α under f,



that is, $\beta = f \circ \alpha$, $\beta : I \to Y$ is a loop based at y_0. Thus, f maps loops based at x_0 into loops based at y_0.

What does f do to equivalence classes of loops? Let α, α' be two loops based at x_0, and let $\alpha \sim \alpha'$ (α, α' are homotopic). Also let $\beta = f \circ \alpha$, $\beta' = f \circ \alpha'$. Are β, β' homotopic?



Yes, because the continuous deformation of α to α' produces a family of curves α_t, $\alpha_0 = \alpha$, $\alpha_1 = \alpha'$, and $f \circ \alpha_t \equiv \beta_t$ is a deformation of β into β'. That is, the composition of two continuous maps is continuous: Here we compose f with the homotopy deforming α to α'.

This means that f induces a mapping between equivalence classes of curves based at x_0 into those based at y_0. Call this mapping K:

$$K: \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0).$$

Every equiv. class at x_0 corresponds to some equivalence class at y_0, i.e. $K[\alpha] = [\beta]$.

The map K is injective. That means that if $[\alpha] \neq [\alpha']$, then $[\beta] \neq [\beta']$, i.e., if α not $\sim \alpha'$ then β not $\sim \beta'$. That is, the only way we can have $[\beta] = [\beta']$ is if $[\alpha] = [\alpha']$. To prove this we may show that $[\beta] = [\beta'] \Rightarrow [\alpha] = [\alpha']$. Since above we showed that $[\alpha] = [\alpha'] \Rightarrow [\beta] = [\beta']$, the result will be $[\alpha] = [\alpha'] \Leftrightarrow [\beta] = [\beta']$, and K is injective.

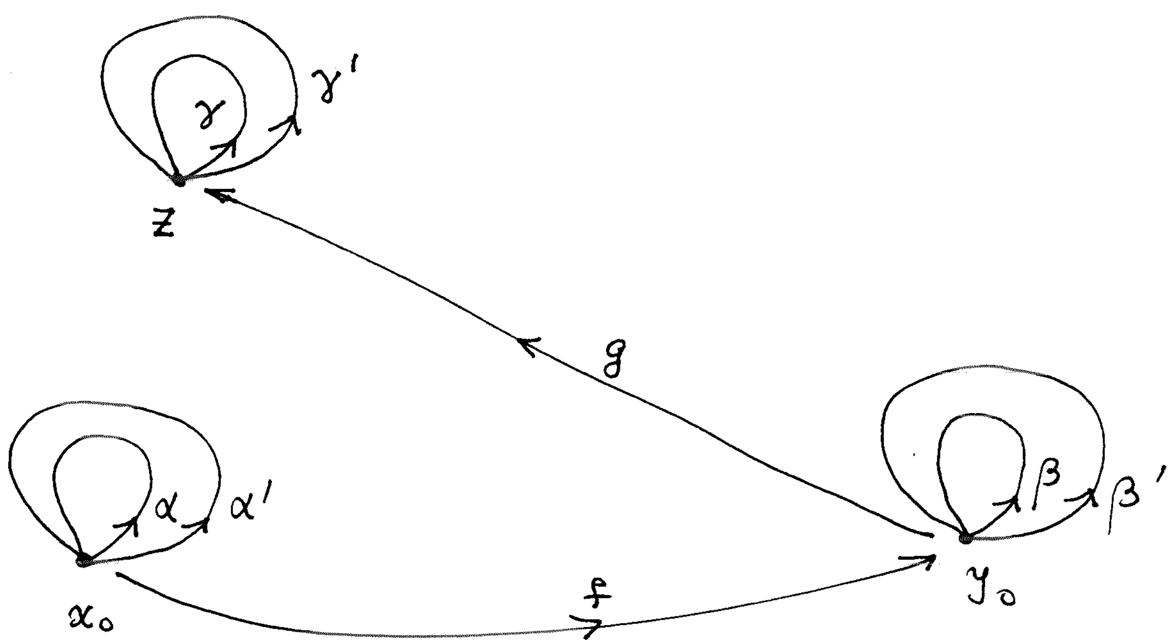
So again assume we have 2 loops α, α' at $x_0 \in X$, and define $\beta = f \circ \alpha$, $\beta' = f \circ \alpha'$, so β, β' are loops at $y_0 = f(x_0) \in Y$. Now, however, assume $\beta \sim \beta'$. We wish to prove that $\alpha \sim \alpha'$.

Use the map g to map β, β' back onto X, call the new loops γ, γ':

$$\gamma = g \circ \beta = g \circ f \circ \alpha$$
$$\gamma' = g \circ \beta' = g \circ f \circ \alpha'$$

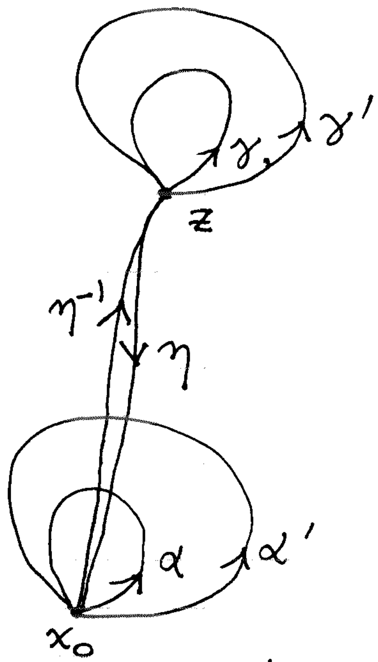
Since g is continuous, and since $\beta \sim \beta'$, we have $\gamma \sim \gamma'$. But γ, γ' are not based at x_0, they are based at a new point call it $z \in X$,

$$z = g(y_0) = (g \circ f)(x_0).$$



We know that $\pi_1(X, z)$ and $\pi_1(X, x_0)$ are isomorphic if we can find a curve connecting z to x_0. In fact, we can, since we know that $g \circ f \sim id_X$. Thus there exists a smooth family of maps $M_t : X \to X$, $t \in [0,1]$, such that $M_0 = g \circ f$ and $M_1 = id_X$. Define a path $\eta : [0,1] \to X$

by $\eta(t) = M_t x_0$. Then $\eta(0) = z$, $\eta(1) = x_0$:



And let η^{-1} run from x_0 to z. Then

$$\eta^{-1} * \gamma * \eta \equiv \delta$$

and

$$\eta^{-1} * \gamma' * \eta \equiv \delta'$$

are loops based at x_0.

In fact, $\delta \sim \delta'$. We see this since we know $\gamma \sim \gamma'$, and if in the definitions of δ, δ' we let γ deform into γ', we get δ deforming into δ'. It's clear from the picture, too.

Now write out δ, δ' in another way:

$$\delta = \eta^{-1} * (g \circ f \circ \alpha) * \eta = \eta^{-1} * (M_0 \circ \alpha) * \eta$$

$$\delta' = \eta^{-1} * (g \circ f \circ \alpha') * \eta = \eta^{-1} * (M_0 * \alpha') * \eta$$

Now let $z(t) = M_t(x_0)$, so $z(t)$ runs along η as t goes from 0 to 1. $z(0) = z$, $z(1) = x_0$. Also let η_t be the curve that starts at $z(t)$ and ends at x_0, following η, i.e., let

$$\eta_t : [0,1] \to X, \quad \eta_t(s) = \eta(t + (1-t)s),$$

so $\eta_t(0) = z(t)$, $\eta_t(1) = x_0$. Thus $\eta_0 = \eta$, and η_1 is the constant path at x_0 (it shrinks to x_0 as $t \to 1$).

Now define

$$\delta_t = \eta_t^{-1} * (M_t \circ \alpha) * \eta_t$$

$$\delta'_t = \eta_t^{-1} * (M_t \circ \alpha') * \eta_t$$

at $t=0$ we have: $\delta_0 = \delta$, $\delta'_0 = \delta'$, at $t=1$ we have $\delta_1 = \alpha$, $\delta'_1 = \alpha'$. Thus, $\delta \sim \alpha$ and $\delta' \sim \alpha'$. But since $\delta \sim \delta'$ we have $\alpha \sim \alpha'$. QED for this part, $\beta \sim \beta' \Rightarrow \alpha = \alpha'$.

Thus $K : \pi_1(X \to x_0) \to \pi_1(Y, y_0)$ is injective.

The remaining steps of the proof are to show that K is surjective (hence bijective), and then finally that it is a group homomorphism (hence an isomorphism).

(Back to deformation retracts.)

we don't talk about deforming spaces (which would require an imbedding space), we talk about deforming maps.

Other examples of same homotopy type spaces:

X	Y
$\mathbb{R}^n$	{one point}

annulus — circle S^1

← These examples show that being of "same homotopy type" allows dimensions of spaces X, Y to be different, no invertible map

If sets are homeomorphic, then they are of same homotopy type, ~~and conversely, this means that~~ (but not the converse).

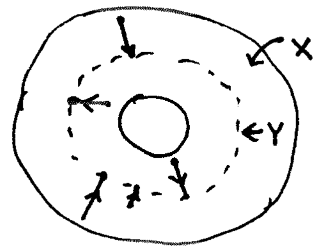
Fact: "Same homotopy type" is an equivalence relation.

Thm: If X and Y are of the same homotopy type, then $\pi_1(X) = \pi_1(Y)$.

We won't prove this thm, but the proof is not very hard, and if you try it you will see the motivation for the definition of "same homotopy type". In any case, this theorem implies that homotopy groups are topological invariants.

So how to tell if spaces are of same homotopy type? May not be easy, but one case arises a lot in practice, occurs when Y is a subset of X. Suppose we have a family of continuous maps parameterized by a deformation parameter $t \in [0,1]$ which is the identity at $t=0$, which maps X onto Y when $t=1$, and which leaves points of Y alone for all t.

Picture, example: X = annulus
Y = circle (dotted)



shown are paths of a point $x \in X$ under the map, as deformation parameter goes from 0 to 1.

Official definition: Let X, Y be topological spaces, $Y \subset X$.
A deformation retract is a map $F: X \times [0,1] \to X$ such that

$F(x,0) = x$ (identity @ $t=0$) | $F(y,t) = y$, $\forall t \in [0,1]$
$F(x,1) \in Y$ (into Y at $t=1$) | (Y invariant, all t).

Fact: If ~~X~~ Y is a deformation retract of X, then X, Y are of same homotopy type. [$f: X \to Y$ is the retraction at $t=1$, $g: Y \to X$ is the inclusion.]

Another Def. If $x_0 \in X$ is the deformation retract of X (special case $Y = \{x_0\}$ = one point), then X is contractible. A contractible space is necessarily connected.
Follows immediately,

Corollary: If X is contractible, then $\pi_1(X) = \{e\}$ (the trivial group).

Def. If $\pi_1(X) = \{e\}$, then X is simply connected.

Before going on, let's get some <u>examples</u> of fundamental groups, obtained by intuition if nothing else.

(1) First, $\pi_1(\mathbb{R}^n) = \{e\}$ (the trivial group), because all loops are contractible (the space is simply connected). This is obvious by drawing pictures,

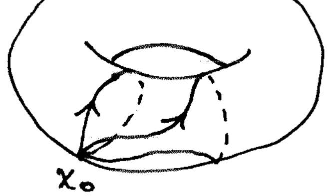
(1a) Note the special case $n=0$, $\mathbb{R}^0$ = one point $= \{0\}$, π_1(one point space) $= \{e\}$.

(2) Next, $\pi_1(S^1) = \mathbb{Z}$, where the integer $n \in \mathbb{Z}$ is the "winding number" of the map. This is intuitively clear if you wrap a rubber band around a cylinder. We consider a more formal argument below.

(3) Next, $\pi_1(S^n) = \{e\}$ for $n > 1$. This is intuitively obvious for S^2 (any closed loop on S^2 can be contracted to a point):

and a similar logic works for S^n for higher n.

(4) Next, $\pi_1(T^2) = \mathbb{Z}^2$ (the 2-torus), as is intuitively clear by drawing pictures,



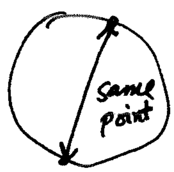
The homotopy class is determined by the two "winding numbers".

This can be proved from the case $\pi_1(S^1) = \mathbb{Z}$ by using the theorem that the fundamental group of the Cartesian product of two arcwise connected spaces is the Cartesian products of the groups,

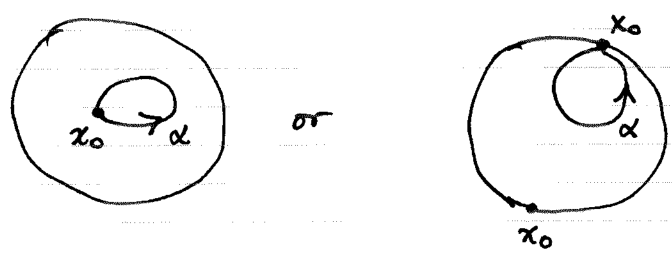
$$\pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y).$$

The same thing shows that $\pi_1(T^n) = \mathbb{Z}^n$ (n winding numbers on an n-torus), since $T^n = S^1 \times \ldots \times S^1$ (n times).
(also mention cylinder $= \mathbb{R} \times S^1$).

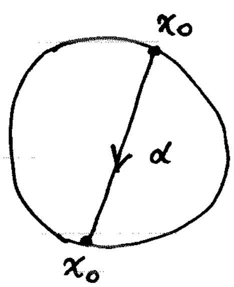
(5) Take the case of $\mathbb{RP}^2$, ~~circle~~ disk with opposite bdry points identified.



Easy to find contractible ~~curves~~ loops:

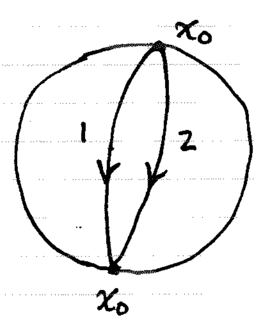


Here is a ~~curve~~ loop that is not contractible:

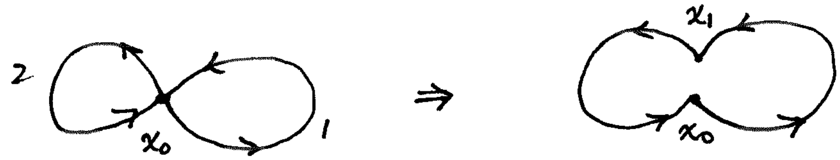


Because you can't bring the two attachment points together (they stay on opposite sides of the bdry under any contin. deformation).

So $[\alpha]$ is a nontrivial element of $\pi_1(\mathbb{RP}^2, x_0)$. Now look at $[\alpha] * [\alpha]$, equiv. class of loops that ~~go~~ are homotopic to traversing α twice. Label the 2 traversals 1, 2 to indicate order, and bow them out slightly to separate them:

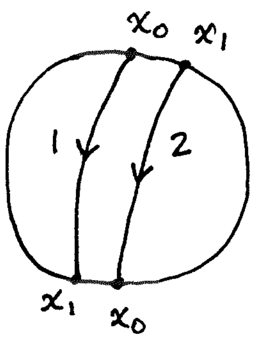


This path starts at x_0, passes through x_0 a 2nd time, then comes back to x_0 a 3rd time. Pull the path away from x_0 on the second encounter, like this:

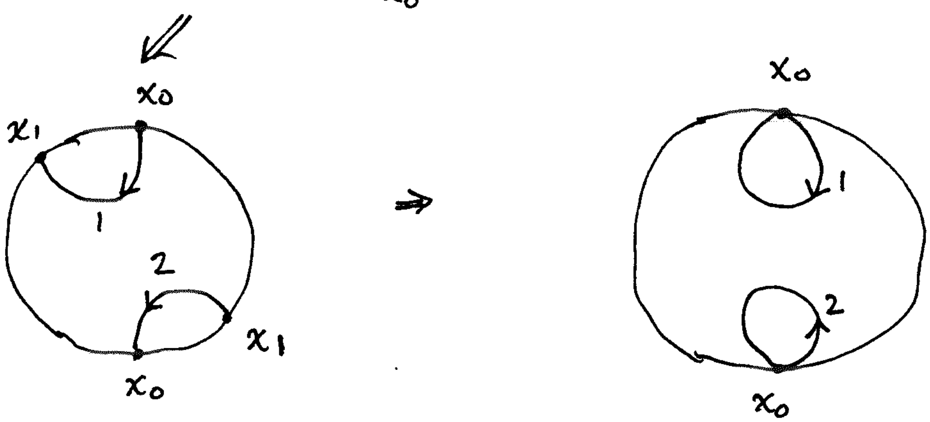


(still leaves a loop based at x_0)

which on $\mathbb{R}P^2$ looks like this:

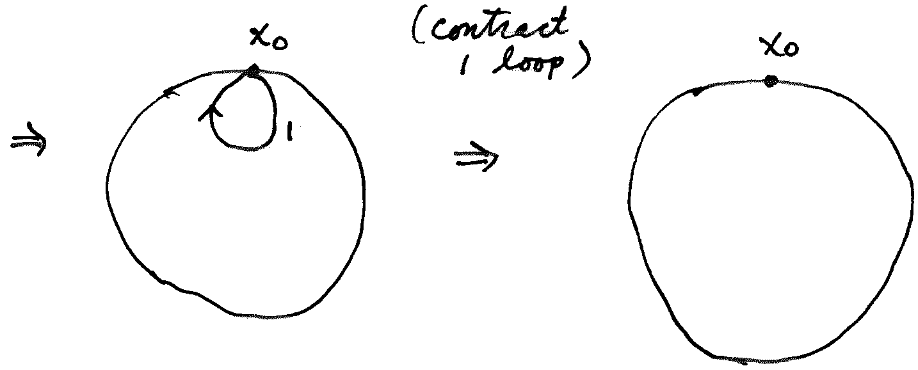


Then deform further by moving x_1 around to meet x_0:



(contract 2 loop)

(contract 1 loop)



(trivial loop at x_0).

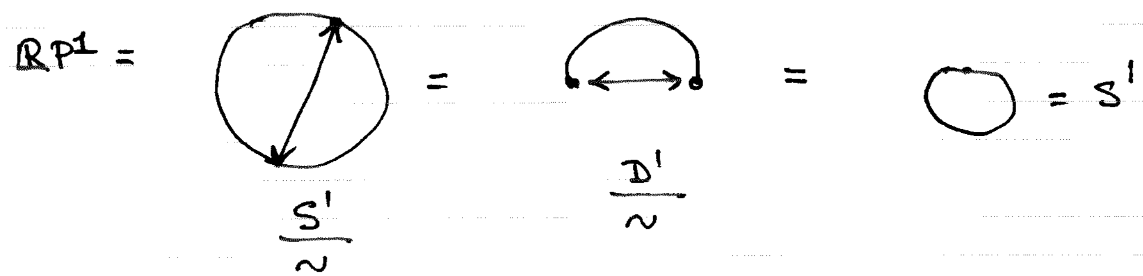
Thus $[\alpha] * [\alpha] = [c]$ = trivial class, contractible.

Hence $\pi_1(\mathbb{R}P^2) = \mathbb{Z}_2$. A similar argument shows that

$\pi_1(\mathbb{R}P^n) = \mathbb{Z}_2$, $n \geq 2$.

This $\mathbb{Z}_2$ is responsible for the line defect (vortex) in a nematic liquid, that can annihilate when it meets another such line defect, leaving behind no defect at all. [It's not that vortices of "opposite charge" are annihilating, rather, there is only one charge and it obeys the rule $1+1=0$.]

While on the subject of $\mathbb{RP}^n$, note special case $n=1$. Recall, $\mathbb{RP}^n$ is the sphere S^n with antipodal points identified; equivalently, it is the n-dimensional disk D^n (= the "northern hemisphere" of S^n) with antipodal points on the boundary identified. (The disk D^n is region $r \le 1$ in n-dim. space $\mathbb{R}^n$; it is the sphere S^{n-1} plus all interior points.) So, for $n=1$, we get a circle with opposite points identified, or D^1, the 1-disk, which is a line segment with ~~opposite~~ endpoints identified:



You might say the final circle is 1/2 as big as the first one.

Now is a good time to comment on the relationship between classical rotations and spin rotations in QM. A classical rotation is an element of $SO(3)$, a linear map of $\mathbb{R}^3$ onto itself that preserves lengths, angles, and cross products. It takes 3 parameters to specify some $R \in SO(3)$ (i.e., $SO(3)$ is a 3-dimensional manifold). These parameters can be specified in various ways. One is the Euler angles (often an ugly choice). Another is the <u>axis-angle</u> parameterization:

(Continued on p. 11).