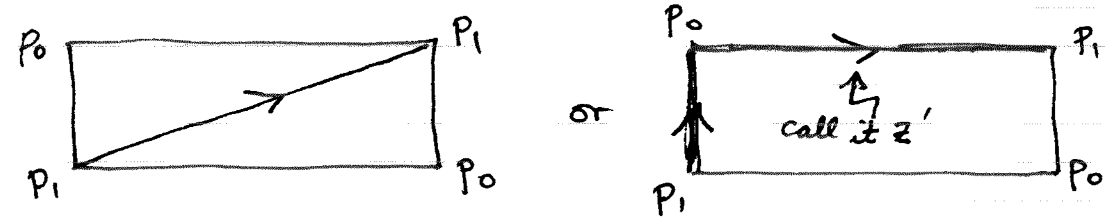
But these have uncancelled internal edges, which you can't get rid of unless you add more triangles to the mix, whereupon you get the whole (Mö), whose boundary is not z. So $z \neq 0$ is a cycle that is not a boundary.
Are there any other, independent ones, that is, 1-cycles that are not boundaries and are not homologous to some multiple of z?
We might try



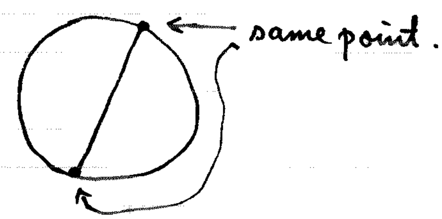
But z' can be deformed into z, so z and z' should be homologous. Indeed, if we calculate we find

$$z - z' = \partial\,(\text{Mö})$$

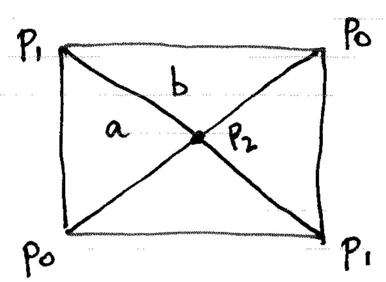
There is only one independent 1-cycle z that is not a boundary, and $[z]$ generates $H_1(K)$:

$$H_1(K) = \{\, n[z] \mid n \in \mathbb{Z} \,\} = \mathbb{Z}.$$

$\boxed{7}$ Now $\mathbb{R}P^2$, which can be realized as disk with opposite points on circumference identified:

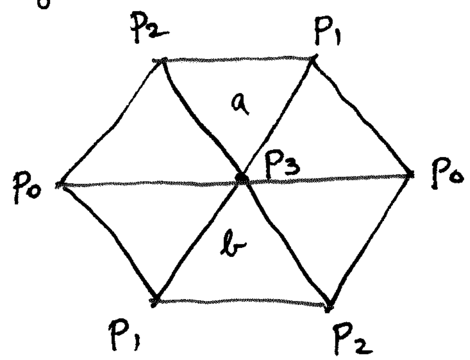


First we have to triangulate.
A square won't work:



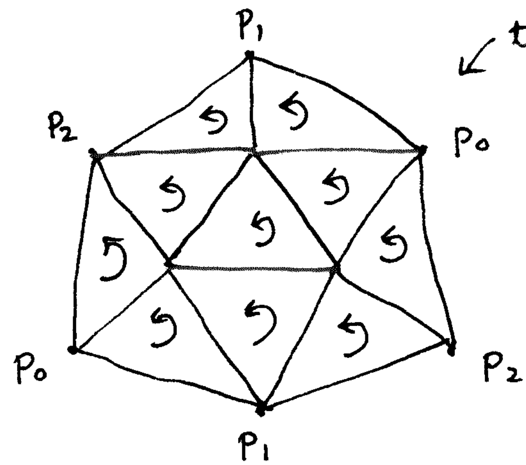
because triangles a, b are identical in K.

So try a hexagon:



Won't work, again because triangles a, b are identical.

So use one internal triangle,



this works.

and orient the triangles as shown.

10 triangles (faces)
15 edges
6 vertices.

$$\left.\begin{array}{l} C_2(K) \cong \mathbb{Z}^{10} \\ C_1(K) \cong \mathbb{Z}^{15} \\ C_0(K) \cong \mathbb{Z}^{6} \end{array}\right\}$$ pretty large dimensions.

$r=0$ trivial, $H_0(K) = \mathbb{Z}$.

$r=2$: As always, $H_n(K) = Z_n(K)$ (here $n=2$) (and cylinder). Are there any 2-cycles (closed surfaces)? Unlike the Möbius strip, $\mathbb{RP}^2$ has no edge, so there are no triangles with edges that are not shared with another triangle. In that sense $\mathbb{RP}^2$ is a "closed" surface. <u>But</u>, the orientations of the triangles are not coordinated, i.e., not all common edges have opposite orientation. To put it another way: if we want to construct a 2-cycle, it must be a linear. comb. of the triangles shown. If it contains any one of the triangles shown, then it must include its ~~neighbo~~ neighbors in order to cancel the internal edges. So it must include the whole set of 10 triangles. Call this $(\mathbb{RP}^2) = \sum \left(\begin{array}{c}\text{ten triangles} \\ \text{w. orientation shown}\end{array}\right)$.

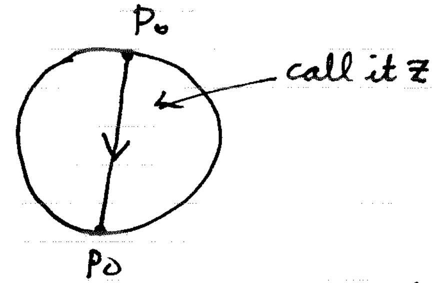
But

$$\partial(\mathbb{R}P^2) = 2\left[(p_0p_1) + (p_1p_2) + (p_2p_0) \right] \neq 0.$$

Therefore there are no 2-cycles (apart from 0), and $H_2(K) = \{0\}$.

<u>Now $r=1$:</u> We need to find 1-cycles that are not boundaries.

An obvious choice is



which can be distorted to run along the triangulation, in fact, we can identify z with $(p_0p_1) + (p_1p_2) + (p_2p_0)$. This is not a boundary (use same logic as on Möbius strip).

Unlike the case of the Möbius strip however, $2z = \partial(\mathbb{R}P^2)$ is a boundary! So $[z]$ is a generator of $H_1(K)$, but it obeys the rule $2[z] = 0$.
By experimenting, we conclude that there are no other 1-cycles, indep. of z and not a boundary. Thus,

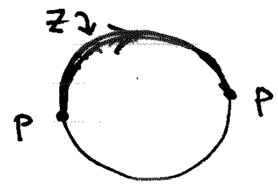
$$H_1(K) = \text{gen}\{[z]\} \cong \mathbb{Z}_2.$$

This is our first example of one of the $\mathbb{Z}_k$ groups in a homology group. This is sometimes described as being due to the "twisting" of the space.

Skip other examples in book (torus, genus g surface Σ_g, Klein bottle).

Now some remarks on what can happen when we allow <u>real</u> coefficients when forming chains. To distinguish the cases, we'll write $H_r(K, \mathbb{Z})$ for the homology group in which chains are restricted to integer coefficients, and $H_r(K, \mathbb{R})$ for real coefficients. Also may consider $H_r(K, \mathbb{Z}_2)$ (these are the most popular choices).

As an illustration, consider the 1-cycle z in $\mathbb{RP}^2$,



Under $\mathbb{Z}$ coefficients, z is not a boundary, but $2z = \partial(\mathbb{RP}^2)$ is. This $[z]$ generates the torsion subgroup $\mathbb{Z}_2$ of $H_1(K, \mathbb{Z})$. But if we allow real coefficients, then z <u>is</u> a boundary, i.e., of the 2-chain $\frac{1}{2}(\mathbb{RP}^2)$. So with $\mathbb{R}$ coefficients, there are no 1-cycles that are not boundaries, and

$$H_1(\mathbb{RP}^2, \mathbb{R}) = \{0\} = \mathbb{R}^0 .$$

More generally, when we compute quotient groups,

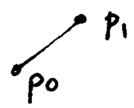
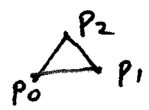
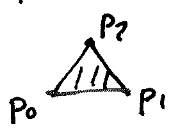
$$H_r(K, \mathbb{R}) = \frac{Z_r(K, \mathbb{R})}{B_r(K, \mathbb{R})}$$

it is always of the form $\dfrac{\mathbb{R}^n}{\mathbb{R}^m}$, $n \geq m$, which $= \mathbb{R}^{n-m}$. There is no torsion subgroup, but the free part has the same dimensionality as in the case $H_r(K, \mathbb{Z})$. That is,

If $\quad H_r(K, \mathbb{Z}) = \mathbb{Z}^f \times \mathbb{Z}_{k_1} \times \dots \times \mathbb{Z}_{k_p}$

then $\quad H_r(K, \mathbb{R}) = \mathbb{R}^f \qquad$ (free part only).

A table.

	K	$H_0(K)$	$H_1(K)$	$H_2(K)$	
1.	$\bullet\ p_0$	$\mathbb{Z}$			
2.	$\bullet\ p_0 \quad \bullet\ p_1$	$\mathbb{Z}^2$			
3.		$\mathbb{Z}$	$\{0\}$ ← no 1-cycles		
4.	 $\cong S^1$	$\mathbb{Z}$	$\mathbb{Z}$.		
5.		$\mathbb{Z}$	$\{0\}$	$\{0\}$ ← no 2-cycles	
$5\frac{1}{2}$.	Tetrahedron $\simeq S^2$	$\mathbb{Z}$	$\{0\}$	$\mathbb{Z}$	← $\exists$ one 2-cycle, the surface S^2 itself.
6.	Möbius	$\mathbb{Z}$	$\mathbb{Z}$	$\{0\}$	
7.	$\mathbb{R}P^2$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\{0\}$	
8.	Klein bottle	$\mathbb{Z}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\{0\}$	
9.	Torus T^2	$\mathbb{Z}$	$\mathbb{Z}\times\mathbb{Z}$	$\mathbb{Z}$	

The example $H_1(K)$ for the Klein bottle shows the general form of homology groups,

$$H_r(K) \cong \underbrace{\mathbb{Z}\times\cdots\times\mathbb{Z}}_{f \text{ factors}} \times \underbrace{\mathbb{Z}_{k_1}\times\cdots\times\mathbb{Z}_{k_p}}_{\text{this part called the torsion group.}}$$

↑ this is the free subgroup.

The dimension of the free part of $H_r(K, \mathbb{Z})$, which is the dimension of $H_r(K, \mathbb{R})$, is called the r-th Betti number of K, denoted

$$b_r(K) = \dim H_r(K, \mathbb{R}).$$

There is an interesting relation between the Euler characteristic and the Betti numbers.

The usual Euler characteristic on a 2D surface S (sphere, torus) is

$$\chi(S) = f - e + v,$$

f = # faces (# 2-simplexes in K)
e = # edges (# 1-simplexes in K)
v = # vertices (# 0-simplexes in K).

On the sphere S^2, $\chi = 2$, independent of the triangulation. The obvious generalization of this is to replace f, e, v etc. by the dimension of the r-th chain group $C_r(K, \mathbb{R})$, which is just the number of independent r-chains in K. So, define

$$\chi(K) = \sum_{r=0}^{n} (-1)^r \dim C_r(K, \mathbb{R}). \qquad (n = \dim K).$$

(These dimensions are not topological invariants.)

Now, ∂_r is a map: $C_r(K) \to C_{r-1}(K)$, so (under $\mathbb{R}$, not $\mathbb{Z}$, so ∂ is a linear map),

$$\dim \ker \partial_r + \dim \operatorname{im} \partial_r = \dim C_r(K, \mathbb{R}).$$

or $$\dim Z_r(K) + \dim B_{r-1}(K) = \dim C_r(K)$$ (over $\mathbb{R}$ understood now.)

In the case $r = 0$, there is no $B_{-1}(K)$, but $Z_0 = C_0$ so we may interpret $\dim B_{-1}(K)$ as zero.

So,
$$\chi(K) = \sum_{r=0}^{n} (-1)^r \left[\dim Z_r(K) + \dim B_{r-1}(K) \right].$$

2nd sum $= \displaystyle\sum_{r=0}^{n} (-1)^r \dim B_{r-1}(K)$

$= \displaystyle\sum_{r=1}^{n} (-1)^r \dim B_{r-1}(K)$ (drop B_{-1} term)

$= -\displaystyle\sum_{r=0}^{n-1} (-1)^r \dim B_r(K)$ (shift index)

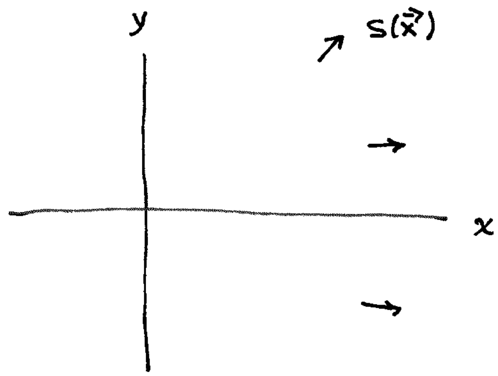
$= -\displaystyle\sum_{r=0}^{n} (-1)^r \dim B_r(K)$ (add $r=n$ term, $\dim B_n(K)=0$).

So whole sum becomes,
$$\chi(K) = \sum_{r=0}^{n} (-1)^r \left[\dim Z_r(K) - \dim B_r(K) \right]$$
$$= \sum_{r=0}^{n} (-1)^r \dim H_r(K) = \sum_{r=0}^{n} (-1)^r b_r(K).$$

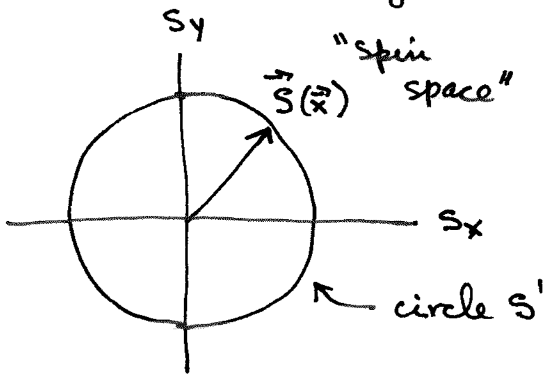
This is the Euler-Poincaré thm, relating the Euler-characteristic to the Betti numbers, manifest topological invariants.

Next we turn to homotopy theory.

Begin with motivation for homotopy theory from CM physics. (But $\exists$ many other applications.) Consider a 2D model of a Heisenberg ferromagnet, treated as a continuum, so we have a spin vector $\vec{S}(\vec{x})$ whose magnitude is fixed, $|\vec{S}| = S = \text{const}$. Assume $\vec{S}$ lies in the plane.

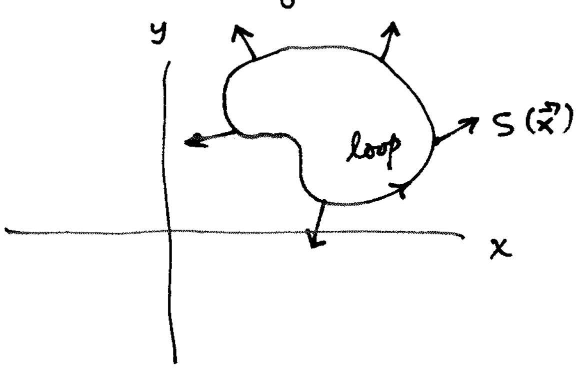


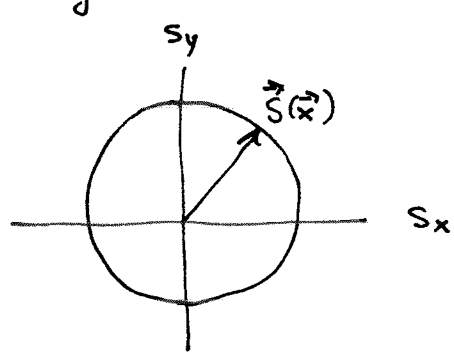
Can regard the spin field as a map $f: X \to S^1$, where X is the region of the plane occupied by the ferromagnet (maybe $X =$ all of $\mathbb{R}^2$, depends on the model). The ~~domain~~ range is S^1 (the circle) because only the direction of $\vec{S}$ can vary, not the magnitude.



The same picture would apply to any 2D vector field in the plane (or some region of it), as long as the vector field did not vanish, (that is, you could look at the direction of the vector field).

Now consider a closed loop in the plane. As we go around the loop, the vector $\vec{S}(\vec{x})$ goes around its circle, returning to where it started.





The number of times $\vec{S}(\vec{x})$ goes around its circle as $\vec{x}$ goes around its loop is the <u>winding</u> <u>number</u> of the field around the loop.

Note that if $\vec{S}(\vec{x}) =$ const (indep. of $\vec{x}$) then the winding # is 0.

Now we can prove that if the winding # is not 0, then there must be a singularity inside the loop. Suppose not. Then we can continuously contract the loop until it is very small. Since we are assuming $\vec{S}$ is continuous (this is what we mean by singularity-free) then over a small loop $\vec{S}$ is nearly constant and the winding # is zero. But the winding number is an integer and cannot change discontinuously. ● So there must be a discontinuity inside the loop.

This would be called a <u>point defect</u> in the plane in CM terminology. The loop provides a map $f: S^1 \to S^1$, where the first S^1 is the loop in the plane and the second is in spin space. The winding number is a characteristic of the map of the circle to itself, also called the <u>Brouwer degree</u> of the map. This number, being an integer, cannot change under continuous deformations of either the loop or the field $\vec{S}(\vec{x})$ itself. Such changes can be thought of as inducing continuous changes of the map $f: S^1 \to S^1$. <u>Homotopy theory</u> studies invariants of maps such as this one under continuous changes.

The range of f (S^1 in this case) is called "order parameter space" in CM applications. Another problem with the same order parameter space is superfluid ^{4}He, where the superfluid is described by a field,

$$\psi(\vec{x}) = A e^{i\varphi}$$

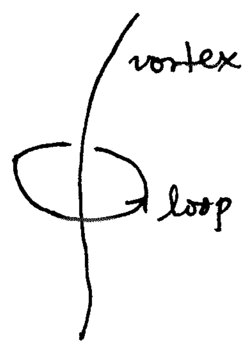
c-number

(ψ is an expectation value of a quantum field, thus it is a classical, complex field on $\mathbb{R}^3$ or some domain $X \subset \mathbb{R}^3$).

The amplitude A is related to the superfluid density which normally is nonzero throughout the volume. The phase φ is related to the fluid velocity via

$$\vec{v} = \frac{\hbar}{m} \nabla \varphi .$$

The phase specifies a point on a circle $e^{i\varphi}$ in the complex plane, so again we may regard order parameter space as a circle and think of a map $f: X \to S^1$ where now $X \subset \mathbb{R}^3$. Or if we restrict f to a loop in X we get a map $f: S^1 \to S^1$ as before, with a winding number. Again, the logic shows that if the winding number is nonzero, then the ~~circle~~ loop must enclose a singularity. This time, however, the domain is in $\mathbb{R}^3$ so the singularity is a line, not a point. It is a line defect, in this case, called a vortex.

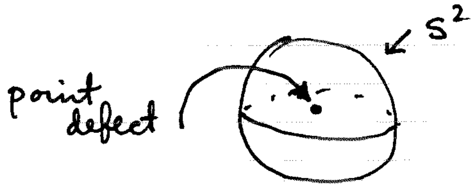


$f: \text{loop} \to S^1$ has winding number

A similar situation occurs in superconductors, when the Cooper-paired spins are in the singlet state (hence Ψ describes a spin 0 boson, like ^4He).

A different type of order parameter space occurs with nematic liquids. These are liquids with long molecules that behave like rigid rods, ~~which~~ tend to align with their neighbors. The aligment is not described by a vector field, however, because the properties of the nematic are the same in both directions, like a double-headed object $\leftrightarrow$.

Thus the order parameter space is $\mathbb{RP}^2$. Nematics support line defects that are studied by examining continuous deformations of maps $f: S^1 \to \mathbb{RP}^2$ (S^1 because a circle goes around a line defect). It turns out that nematics also support point defects that are studied by surrounding the point with a closed surface, i.e., one homeomorphic to S^2, and studying invariants of maps $f: S^2 \to \mathbb{RP}^2$.

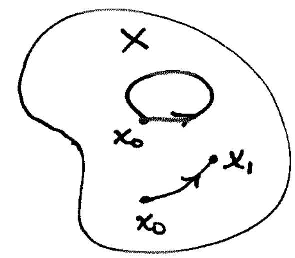


There are other kinds of order parameter spaces. For spins in 3D, the spin vector is identified with a point on S^2. There are no line defects, but point defects are possible and lead us to study maps $S^2 \to S^2$. Another example is superfluid ^{3}He in the dipole locked phase, where the order parameter can be thought of as two vectors (nonzero), locked to be orthogonal to one another. The orientation of such an object is specified relative to a standard orientation, by means of a rotation $\in SO(3)$. Hence order parameter space is $SO(3)$, which is homeo. to $\mathbb{RP}^3$ (as it turns out).

To study defects we have looked at maps from $\underset{\text{line}}{S^1}$ or $\underset{\text{point (in } \mathbb{R}^3)}{S^2}$ to order parameter space (S^1, $\mathbb{RP}^2$, S^2, $SO(3)$, ...). A map from S^3 to order parameter space occurs when we want to study <u>singularity-free</u> field configurations ~~that~~ in $\mathbb{R}^3$ that satisfy the constraint of having a fixed asymptotic value as $r \to \infty$. We just "compactify" $\mathbb{R}^3$ into S^3 by introducing the "point at ∞" (where the field has a fixed value), and study maps $f: S^3 \to$ O.P.S. such that f(pt. at ∞) = given value. For example, the constant map is one such field configuration, but (depending on the O.P.S.) there may be others which cannot be reached by continuous deformation from the constant case. Again we are interested in properties of maps $f: S^n \to M$ (some manifold = O.P.S.) that are invariant under contin. deformations.

We take the preceding as motivation for considering continuous deformations of maps : $S^n \to M$ = some manifold. First take case S^1.

Let $I = [0,1]$.

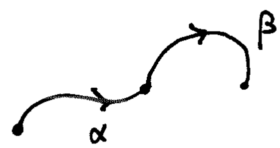


A <u>path</u> is a map $f: I \to X$ = some topological space

A <u>loop</u> is a path such that $f(0) = f(1) = x_0$ = "<u>base point</u>"

Note, path can self-intersect, or even be just a point (the constant path).

Now, basic properties of paths and loops. First, we can <u>multiply</u> paths if the endpoints match. This is just <u>catenation</u>.



$= \alpha * \beta$.

Let $\alpha, \beta : [0,1] \to X$, let $\alpha(1) = \beta(0)$. Define:

$$(\alpha * \beta)(s) = \begin{cases} \alpha(2s), & 0 \le s \le 1/2 \\ \beta(2s-1), & \frac{1}{2} \le s \le 1 \end{cases}.$$

Notice order: traverse α first, β second.

Note that loops based at a common point can always be multiplied.

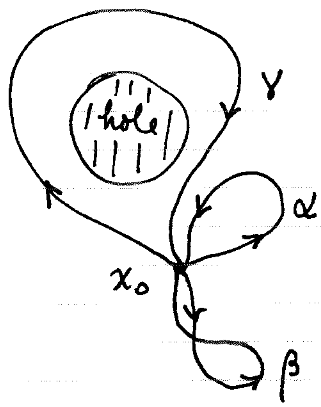
$\alpha * \beta$ meaningful.

Def: Let $\alpha : [0,1] \to X$ be a path. Define the <u>inverse path</u> by

$$\alpha^{-1} : [0,1] \to X, \qquad \alpha^{-1}(s) = \alpha(1-s).$$

The inverse path just traverses the original path in the reverse order.

Now we define <u>homotopic equivalence</u>, specializing to the case of ~~and~~ loops based at a point. Idea:



α is homotopically equivalent to β, but not to γ (they cannot be continuously deformed into one another).

Let $\alpha, \beta : \underbrace{[0,1]}_{=I} \to X$ be loops based at $x_0 = \alpha(0) = \alpha(1) = \beta(0) = \beta(1)$.

Then α is said to be <u>homotopic</u> to β if $\exists$ a continuous function

$F : I \times I \to X$ (think, $F(s,t)$, $s =$ parameter of path, $t =$ deformation param.)

such that

$$F(s,0) = \alpha(s) \qquad \text{and} \qquad F(0,t) = F(1,t) = x_0.$$
$$F(s,1) = \beta(s)$$

Then F (the deformation map) is said to be the <u>homotopy</u>.

Basic fact is, "homotopic to" is an equivalence relation. To prove this, have to show 3 things: