Examples of Abelian groups include:

$$\mathbb{R}^n, \; S^1, \; \mathbb{Z}, \; \mathbb{Z}_k$$

- $\mathbb{R}^n$: vector space
- S^1: $\{e^{i\theta}\}$.
- $\mathbb{Z}_k$: integers modulo k $= \{0, 1, \ldots, k-1\}$.

We will only be interested in <u>discrete</u> Abelian groups, which excludes things like $\mathbb{R}^n$ and S^1.

If G is ~~an~~ a discrete Abelian group and $x_1, \ldots, x_r$ are elements of G such that any $g \in G$ can be written in the form,

$$g = \sum_{i=1}^{r} n_i x_i, \qquad n_i \in \mathbb{Z},$$

then G is said to be <u>generated</u> by the $\{x_i\}$ and the $\{x_i\}$ are said to be the <u>generators</u>. If $r < \infty$, then we say that G is <u>finitely generated</u>. For homology theory we only need finitely generated Abelian groups. Notice that so far we're not saying that the generators are minimal in number (and in any case they are not unique).

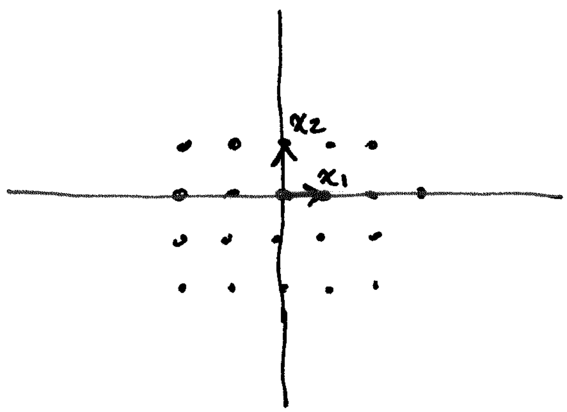
Either the group elements $\sum_{i=1}^{r} n_i x_i$ for $n_i \in \mathbb{Z}$ are all unique, or there is some duplication (some group elements can be represented as a "linear combination" of the generators in more than one way). In the former case we say that the group is <u>freely generated</u>, and that G is a <u>free Abelian group of rank r</u>. In this case, every $g \in G$ can be <u>uniquely</u> represented as

$$g = \sum_{i=1}^{r} n_i x_i,$$

and G is isomorphic to $\mathbb{Z}^r$, $G \cong \mathbb{Z}^r$, $g \mapsto (n_1, \ldots, n_r)$.

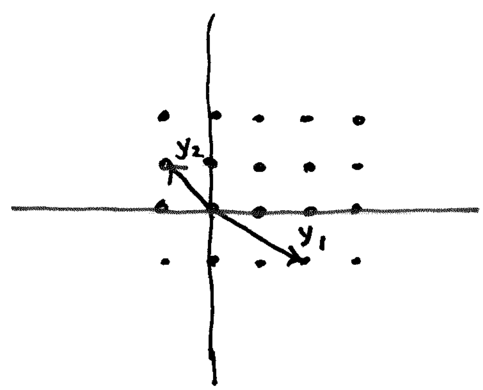
In effect, there <u>is</u> only one free Abelian group of rank r, it is $\mathbb{Z}^r$, and it can be visualized as the integer lattice in r-dimensional $\mathbb{R}^r$.

About the non-uniqueness of the generators. Take the case $r=2$ for illustration. Represent the generators x_1, x_2 as unit vectors in the plane, and the group as the lattice.



The "basis" (x_1, x_2) spans the lattice, but it is not unique. We could use

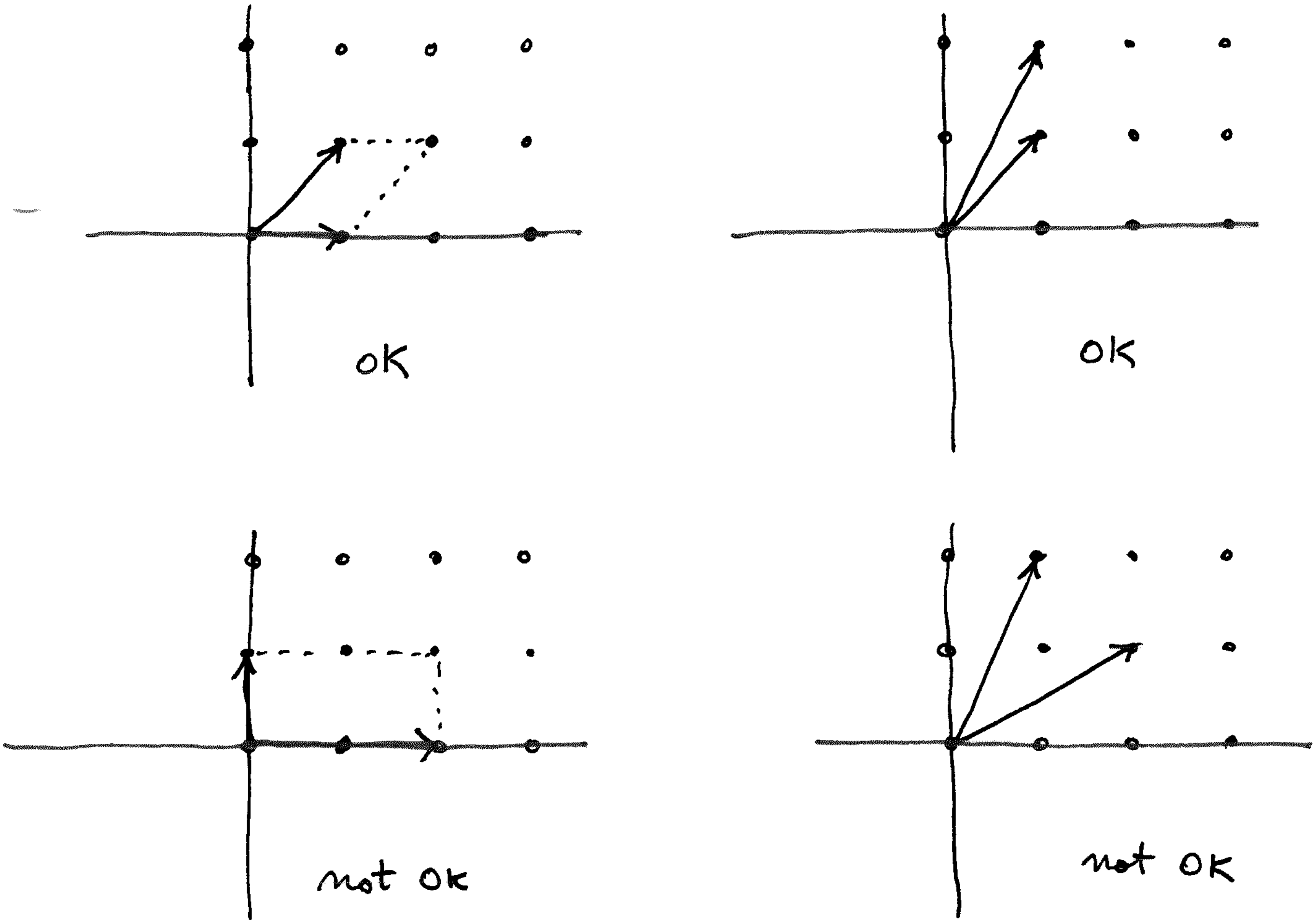
$$y_1 = 2x_1 - x_2$$
$$y_2 = -x_1 + x_2$$



or $$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

↑ note det = +1.

But not any linear combination will work. The new basis (y_1, y_2) must span a cell that does contain any points inside or on the boundary, except at the 4 corners (= avg. of one point per cell, since each corner is shared among 4 cells).



The requirement for a valid change of basis is that the $r \times r$ matrix M,

$$\begin{pmatrix} y_1 \\ \vdots \\ y_r \end{pmatrix} = M \begin{pmatrix} x_1 \\ \vdots \\ x_r \end{pmatrix}$$

must consist of integers and must have an inverse M^{-1} that consists of integers. This means $M \in GL(r, \mathbb{Z})$. It also means that $\det M = \pm 1$.

Notice if we have a freely generated Abelian group (of rank r) generated by $(x_1, \dots, x_r)$, then the only way $\sum_{i=1}^{r} n_i x_i = 0$ is when $n_i = 0$, $\forall i$. Then we may borrow terminology from linear algebra and say that $(x_1, \dots, x_r)$ are <u>linearly</u> <u>independent</u>.

To handle the general case (free or not free) of a finitely generated Abelian group, let G = the group, $\{x_1, \ldots, x_r\}$ a set of generators, and consider the map

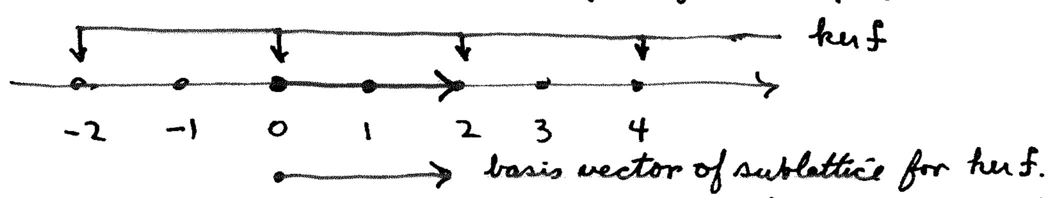
$$f: \mathbb{Z}^r \to G: (n_1, \ldots, n_r) \mapsto \sum_{i=1}^{r} n_i x_i .$$

This map is onto, $\operatorname{im} f = G$, by the definition of "generators".
The condition that the group is free is precisely the condition $\ker f = \{(0, \ldots, 0)\}$, that is, $\ker f$ is the trivial subgroup of $\mathbb{Z}^r$ containing the identity.
Then

$$G \cong \frac{\mathbb{Z}^r}{\ker f} \cong \mathbb{Z}^r, \quad \text{same conclusion as above.}$$

But if the group is not free, then $\ker f$ contains more than the identity element. In fact, it must be a sublattice of $\mathbb{Z}^r$, since it's closed under addition.

As an example, consider the case $r = 1$, so only one generator x. An Abelian group G with one generator is called cyclic. If the group is free, then $G \cong \mathbb{Z}$. But suppose for example, that $2x = 0$.



Then $\ker f$ is the set $(\ldots, -2, 0, 2, 4, \ldots)$, the sublattice spanned by (2). Call this subgroup $2\mathbb{Z}$.
↳ also sublattice.

[More generally, $k\mathbb{Z}$ is the set $\{kn \mid n \in \mathbb{Z}\}$ for $k \geq 1$]

Then $$G = \frac{\mathbb{Z}}{2\mathbb{Z}} \cong \mathbb{Z}_2 = \{0, 1\}, \quad \text{integers modulo 2.}$$

More generally, $\frac{\mathbb{Z}}{k\mathbb{Z}} \cong \mathbb{Z}_k = \{0, 1, \dots, k-1\}$ = integers modulo k.

We see that a cyclic group either contains an ∞ number of elements, in which case it is isomorphic to $\mathbb{Z}$, or else it contains a finite number $k \geq 1$ elements, in which case it is isomorphic to $\mathbb{Z}_k$.

Now we quote the facts (without proof) for the case of arbitrary r.

As above, G = finitely generated Abelian group with generators $\{x_1, \dots, x_r\}$, and $f: \mathbb{Z}^r \to G : (n_1, \dots, n_r) \mapsto \sum_{i=1}^{r} n_i x_i$. Again, $\operatorname{im} f = G$.

Fact 1. Any subgroup of $\mathbb{Z}^r$ is a sublattice of $\mathbb{Z}^r$ which can be spanned by some set of integer vectors (lin. independent) (elements of $\mathbb{Z}^r$), call them $\{y_1, \dots, y_p\}$, $p \leq r$. In particular, $\ker f$ can be written,

$$\ker f = \left\{ \sum_{i=1}^{p} m_i y_i \;\middle|\; m_i \in \mathbb{Z} \right\}.$$

This is the general form of a finitely generated Abelian group.

Fact 2. $$\frac{\mathbb{Z}^r}{\ker f} \cong \underbrace{\mathbb{Z}_{k_1} \times \mathbb{Z}_{k_2} \times \dots \times \mathbb{Z}_{k_p}}_{p \text{ factors}} \times \underbrace{\mathbb{Z} \times \dots \times \mathbb{Z}}_{r-p \text{ factors}} \cong G.$$

$$k_i \geq 1, \quad i = 1, \dots, p.$$

Note that $\mathbb{Z}_1$ (the case $k=1$) is just the trivial group $\{0\}$ (the cyclic group with one element); if this occurs in the list it can be dropped. Note also that we never had to say that the generators $\{x_1, \dots, x_r\}$ were "minimal"; if a smaller set of generators would work, then there will appear $\mathbb{Z}_1$ factors in the final product.

Now we turn to the machinery needed to formalize the idea of a polyhedron.

Def. A <u>r-simplex</u> is a region of $\mathbb{R}^m$ $(m \geq r)$ specified by $r+1$ "geometrically independent" points, $p_0, p_1, \ldots, p_r$, denoted $\sigma = \langle p_0 p_1 \cdots p_r \rangle$, and defined by

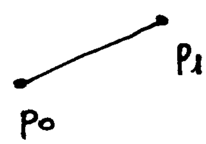
$$\sigma = \langle p_0 p_1 \cdots p_r \rangle = \left\{ \sum_{i=0}^{r} c_i p_i \;\middle|\; \sum_{i=1}^{r} c_i = 1, \quad c_i \geq 0 \right\}.$$

Coefficients $\{c_i\}$ are <u>barycentric coordinates</u>. The <u>order</u> of the points in $\langle p_0 \cdots p_r \rangle$ is unimportant (any permutation represents the same simplex.)

<u>Examples:</u>

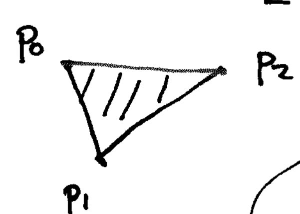
$\sigma_0 = \langle p_0 \rangle$ = a point

$\sigma_1 = \langle p_0 p_1 \rangle$ = an edge



$x = c_0 p_0 + c_1 p_1$
$= c_0 p_0 + (1 - c_0) p_1$
$0 \leq c_0 \leq 1.$

$\sigma_2 = \langle p_0 p_1 p_2 \rangle$ = a face (triangle)



Here, "geometrically independent" means $p_0 p_1 p_2$ do not lie on a line: the triangle has positive area.

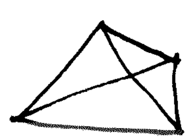
Def. Let $\sigma_r = \langle p_0 \ldots p_r \rangle$ = a simplex

$\sigma_q = \langle p_{i_0} \cdots p_{i_q} \rangle$ = a q-face of σ_r, $(q \leq r)$

where $\{p_{i_0}, \ldots p_{i_q}\}$ is a subset of $\{p_0, \ldots, p_r\}$.

Denote $\sigma_q \leq \sigma_r$.

<u>Example:</u>



σ_3 = tetrahedron = 3-simplex

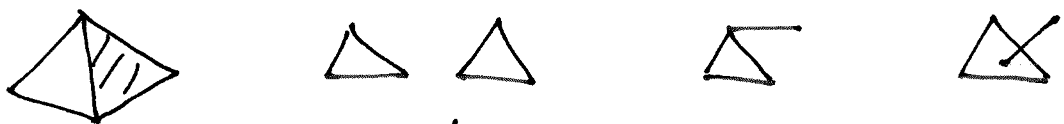
4 2-faces
6 1-faces
4 0-faces

$\#$ q-faces $= \binom{r+1}{q+1}$

Def. A __simplicial complex__ is a set $K = \{\sigma_i\}$ of simplexes such that:

1) If $\sigma \in K$ then K also contains all the faces of σ
2) If $\sigma, \sigma' \in K$ then either $\sigma \cap \sigma' = \phi$ or σ and σ' intersect in a common face.

Property 2 means that simplicial complexes are "nicely fitted together"



ok ok ok not ok

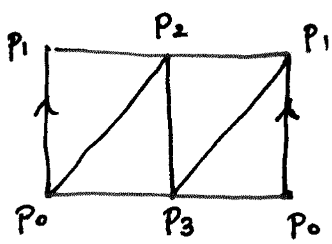
Def. A __polyhedron__ $|K|$ is the set

$$|K| = \bigcup_{\sigma_i \in K} \sigma_i .$$

Def. A topological space X is __triangulable__ if $\exists$ a polyhedron $|K|$ and homeomorphism $f: |K| \to X$.

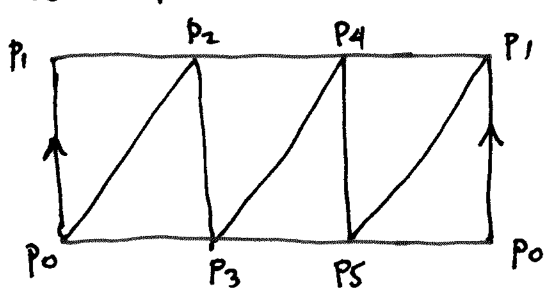
__Example:__ A cylinder (square with 2 sides identified,).

This



doesit work, because $|K|$ is not homeomorphic to the cylinder (it is two rectangles superimposed, i.e., one rectangle).

Insead, use



6 triangles.

__Note:__ the triangulation is __not unique__, because we can always __refine it__ (add more simplices).

$f:$