Leads to definition, $\tilde{f}: W \to V$, the <u>adjoint</u> of f, by

$$\tilde{f} = g^{-1} f^* G \;(= g^{-1} \circ f^* \circ G).$$

This is equivalent to

$$g(\tilde{f}w, v) = G(w, fv), \quad \forall\, v \in V,\ w \in W,$$

or

$$\langle \tilde{f}w, v\rangle_g = \langle w, fv\rangle_G,$$ which should look familiar.

(exercise to show this).

Now what changes when you go to metrics on <u>complex</u> vector spaces.

Now $g: V \times V \to \mathbb{C}$, such that:

1) g is linear in 2nd operand, and <u>anti-linear</u> in the first,

$$g(c_1 u_1 + c_2 u_2, v) = \bar{c}_1\, g(u_1, v) + \bar{c}_2\, g(u_2, v)$$
$$g(u, c_1 v_1 + c_2 v_2) = c_1\, g(u, v_1) + c_2\, g(u, v_2)$$

2) $g(v,v) = \text{real}, > 0, \quad \forall\, v \in V$

$g(v,v) = 0$ iff $v = 0$

3) $g(u,v) = \overline{g(v,u)}$.

(overbar = complex conjugate).

The associated mapping $g: V \to V^*$ is defined as before,

$$u \mapsto g_u, \qquad g_u(v) = g(u,v) = \langle u, v\rangle,$$

but it is now an <u>antilinear</u> map (point missed by Nakahara).

The adjoint is defined as above, $\tilde{f} = g^{-1} f^* G$. Note that it is <u>linear</u>, since g^{-1} and G are antilinear. (Usual notation in QM, $\tilde{f} = f^+$).

Note: usually in QM when we talk about the adjoint of a linear operator, we are thinking of the case $W=V$, so $G=g$, and so $\tilde{f} = g^{-1} f^* g$, and

$$\langle \tilde{f} u, v \rangle = \langle u, f v \rangle.$$

Next, tensors. ($K = \mathbb{R}$ here).

A tensor T of type (p,q) is a multilinear map,

$$T : \underbrace{V^* \times \dots \times V^*}_{p \text{ times}} \times \underbrace{V \times \dots \times V}_{q \text{ times}} \to \mathbb{R}$$

Examples:

A covector $\alpha \in V^*$ is a tensor of type $(0,1)$, since $\alpha : V \to \mathbb{R}$.

A vector $v \in V$ is considered a tensor of type $(1,0)$, since it can be considered a map $v : V^* \to \mathbb{R} : \alpha \mapsto \alpha(v) \equiv v(\alpha)$.

$g : V \times V \to \mathbb{R}$ is a tensor of type $(0,2)$

$g^{-1} : V^* \times V^* \to \mathbb{R}$ " " " " " $(2,0)$

etc.

There are 2 operations on tensors, the tensor product and the contraction.

Let $\mu =$ tensor of type (p,q)
$\nu =$ " " " (r,s)

then $\mu \otimes \nu =$ " " " $(p+r, q+s)$.

Def:
$$\mu \otimes \nu (\alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_r ; v_1, \dots, v_q, u_1, \dots, u_s)$$
$$= \mu(\alpha_1, \dots, \alpha_p ; v_1, \dots, v_q)\, \nu(\beta_1 \dots, \beta_r ; u_1, \dots, u_s).$$

The <u>contraction</u> takes a tensor of type (p,q) and produces one of type $(p-1,q-1)$. It does not require a metric for its definition. We illustrate in case of contraction on first slots. Let μ = a tensor of type (p,q).

$$(\text{contracted}\,\mu)(\alpha_2,\dots,\alpha_p;\, v_2,\dots,v_q) = \sum_i \mu(e^{i*}, \alpha_2,\dots,\alpha_p;\, e_i, v_2,\dots,v_p)$$

where $\{e_i\}$ is a basis in V and $\{e^{i*}\}$ is the dual basis in V^*.

The contraction is independent of the basis (but it does depend in general on which slots are contracted).

Now some background on spaces, manifolds, topology, etc. First some overview of the hierarchy of spaces you get as you add more and more structure.

At the most primitive level, a "space" is just a set of objects we call "points" without any additional structure.

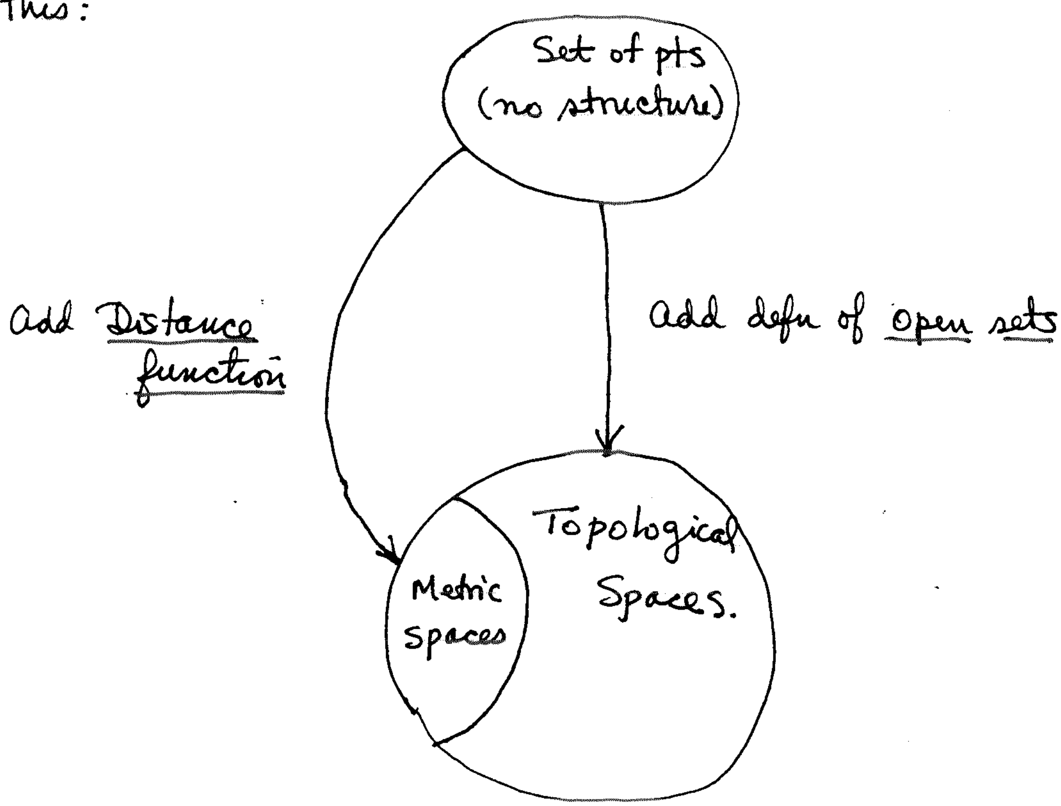
To this we may add the structure required to give the space a <u>topology</u>. Intuitively, topology tells you something about which points are "close" to one another, so that you can think about "neighborhoods" of points and how different neighborhoods fit together to make the whole space.

Naively, it would seem that the concept of "closeness" requires a definition of "distance" between points, but this is not so. Instead, all one needs is a definition of the <u>open</u> <u>subsets</u> of the space (if it is to be a topological space). Here is the official definition:

> A <u>topological space</u> is a set X plus a set $\{U_i\}$ of subsets, the "open subsets", such that
>
> 1) Both X and ϕ (the empty set) are in $\{U_i\}$
> 2) The union of any number (finite or infinite) of open subsets is open (i.e., it is in $\{U_i\}$)
> 3) The intersection of any finite number of open sets is in $\{U_i\}$.

We won't work at the level of this definition very much, but it is worthwhile to have it.

On the other hand, if we take our space X and introduce a distance function, then we obtain a metric space, and open sets can be defined in the usual way using the concept of distance. For example, an "open ball" centered at x_0 is the set of all points x such that $d(x_0, x) < r$ for some radius $r > 0$, where $d(\,,\,)$ is the distance function. Thus, all metric spaces automatically become topological spaces (they form a subset of all topological spaces). (The distance function is not to be confused with a metric tensor; the latter can be used to construct a distance function, but that is just a special case.) Thus we have a diagram like this:



The distance function has to satisfy certain requirements.

But since topological matters can be expressed purely in terms of the open sets (this is the more primitive concept), we use that in developing topology, not distances.

For example, let $f: X \to Y$ be a map between two topological spaces. Then we say that f is <u>continuous</u> if the inverse images of open sets (in Y) are open sets (in X). This is the topological definition of continuity. It coincides with usual ε and δ definition in the case of a metric space, but is more general.

The definition of topology leaves open the possibility that a given set X can be endowed with a topology in more than one way, ~~and~~ (that is, you define open sets in more than one way), and this is true.

In the case of $X = \mathbb{R}$, the <u>usual topology</u> is given by defining open sets to be open intervals and their unions and finite intersections. This is the same as the topology given by the distance function $d(x,y) = |x-y|$, $x, y \in \mathbb{R}$. The usual topology on $\mathbb{R}^n$ is defined similarly. In this course we will only use the usual topology. We can also define the usual topology on subsets of $\mathbb{R}^n$, which covers all the finite dimensional spaces we will use in this course.

The usual topology has the <u>Hausdorff</u> property. See book for definition. We will use the usual topology, so all our spaces will have the Hausdorff property (usual in physical applications).

Let's look at some of the structure that exists at the level of a topological space.

Let X = a topological space.

Def. A subset $A \subset X$ is <u>closed</u> if the complement $X - A$ is open.

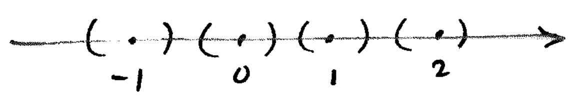
Also define <u>closure</u>, <u>interior</u> (see text).

Definition of <u>continuity</u> given above.

Now compactness, a property that is a little mysterious if you've never taken a course in topology.

Def. A subset $A \subset X$ of a topological space is <u>compact</u> if every open cover contains a finite subcover. (An open cover is a set of open sets whose union contains A.).

<u>Example</u>, $\mathbb{Z} \in \mathbb{R}$.



$\mathbb{R}$

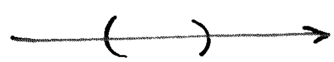
$X = \mathbb{R}$
$A = \mathbb{Z}$.

Open cover has ∞ # of open intervals, surrounding each integer. Remove any one and $\mathbb{Z}$ is no longer covered. Therefore $\mathbb{Z}$ is not compact in $\mathbb{R}$.

Important thm. for compactness in $\mathbb{R}^n$ (Heine-Borel):

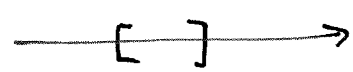
$\boxed{\text{Thm. A subset of } \mathbb{R}^n \text{ is compact if and only if it is closed and bounded.}}$

<u>Examples</u>: 1) Open interval in $\mathbb{R}$



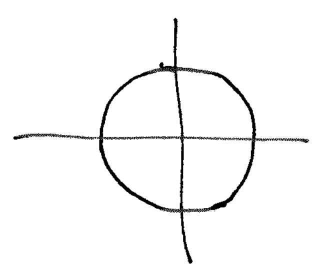
noncompact.

2) Closed " " "



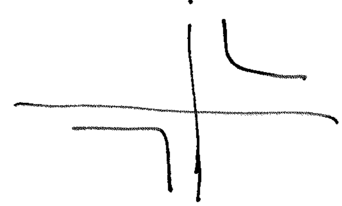
compact.

3) S^1 (circle) in $\mathbb{R}^2$:



compact.

4) Hyperbola in $\mathbb{R}^2$



noncompact.

Mention impact compactness has on group representation theory, e.g.

compact groups such as $SU(n)$, $SO(n)$, etc.

v.s. noncompact ones like Lorentz, $GL(n, \mathbb{R})$, $Sp(2n)$ etc.

Now, Connectedness. There are 2 kinds of notions, connected and arc-wise connected, which are the same in most physical applications. Idea corresponds to intuitive notion of connectedness. Official defns:

A topological space X is connected if it cannot be written as the disjoint union of two open sets.

It is arcwise connected if for $\forall\ x, y \in X$, there exists a continuous curve in X connecting x and y (i.e., a continuous map: $[0,1] \to X$ with $f(0) = x$ and $f(1) = y$.)

Finally, X is simply connected if every closed loop can be continuously contracted to a point.

Now we turn to the principal notion of topological equivalence, namely, the homeomorphism.

> Def. Given two topological spaces X and Y. A map $f: X \to Y$ is said to be a homeomorphism if it is continuous and possesses a continuous inverse $f^{-1}: Y \to X$. ~~then sp~~ If such an f exists, X and Y are said to be homeomorphic.

Nakahara discusses this concept in the framework of a continuous deformation of one space into another (the coffee mug to the doughnut, for example). Continuous deformation, however, requires an embedding space (in which X and Y are subsets), and this plays no part in the definition of homeomorphism.

A central problem of topology is to classify topological spaces, to within a homeomorphism. One can easily show that the relation "homeomorphic" is an <u>equivalence relation</u>, thus topological spaces are divided into equivalence classes.

It is not known how to classify all equivalence classes of topological spaces. That is, given two spaces, it may not be easy to show that they are homeomorphic, apart from finding the homeomorphism that connects them. However, it may be relatively easy to show that they are not homeomorphic, by using topological invariants.

A <u>topological invariant</u> is a quantity or characteristic that is invariant under homeomorphisms. Thus, if two spaces have different invariants, they are not homeomorphic.

See book for more on topological invariants, Euler characteristic.

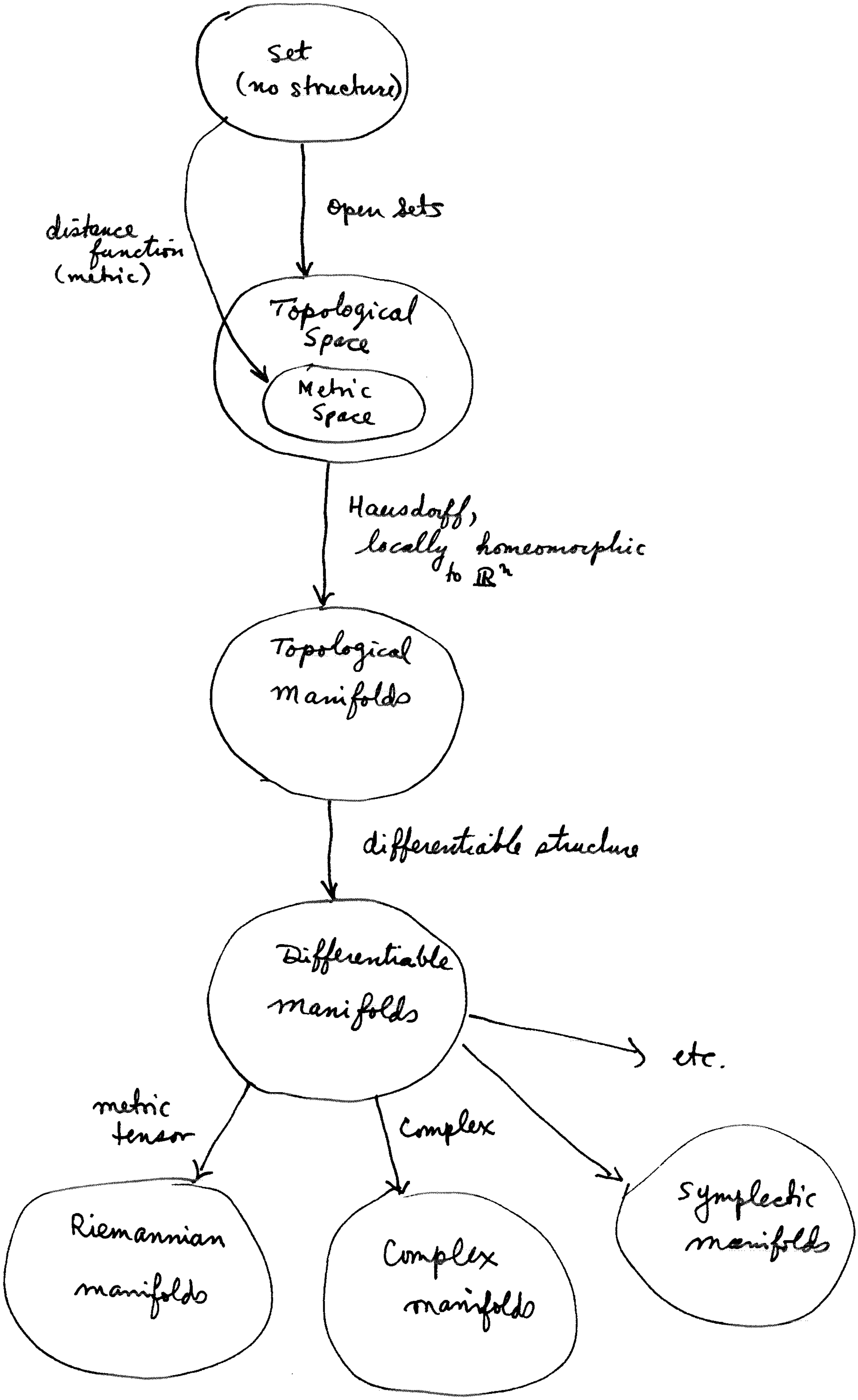
Examples of topological invariants:

1. Compactness
2. Connectedness
3. Simply-Connectedness

(and many more)

Now we restrict topological spaces further by requiring the Hausdorff property, and requiring that the space be "locally homeomorphic to $\mathbb{R}^n$"! The result is a <u>topological manifold</u>. We will deal exclusively with topological manifolds. At this level we can talk about continuity, but not differentiability. The latter requires additional structure (later in course) to turn a topological manifold into a <u>differentiable manifold</u>.

Finally, a differentiable manifold may be given additional structure (such as a metric tensor) to get a <u>Riemannian manifold</u>, <u>complex manifold</u>, <u>symplectic manifold</u>, etc.

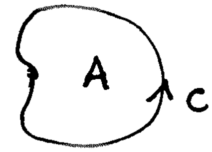

Set
(no structure)
Open sets
distance function (metric)
Topological Space
Metric Space
Hausdorff, locally homeomorphic to $\mathbb{R}^n$
Topological manifolds
differentiable structure
Differentiable manifolds
etc.
metric tensor
Complex
Riemannian manifolds
Complex manifolds
Symplectic manifolds

Turn now to a motivation for <u>homology theory</u>, ch. 3 of Nakahara. Nakahara's presentation starts out rather technically, so motivate some first.

Both C and A have an <u>orientation</u>.

Consider Stokes' Theorem in 3D ($\mathbb{R}^3$):

$$\int_C \vec{F} \cdot d\vec{r} = \int_A \nabla \times \vec{F} \cdot d\vec{a}$$



Curve C is <u>boundary</u> of area A.

C, A examples of <u>chains</u> (defined later).

$$\left.\begin{aligned} C &= 1\text{-chain} \\ A &= 2\text{-chain} \end{aligned}\right\} \text{ things you integrate over.}$$

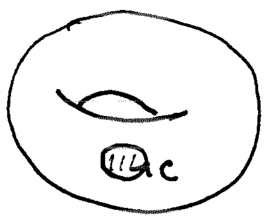
Write,

$$C = \partial A$$

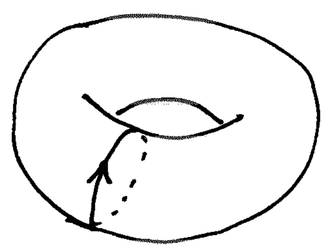
↑ <u>boundary operator</u>

C is a <u>closed</u> <u>curve</u>, called a <u>cycle</u> (or 1-cycle).

On $\mathbb{R}^3$, every closed curve (1-cycle) is the boundary of some 2D region A. But on other spaces this is not true. The 2-torus, for example

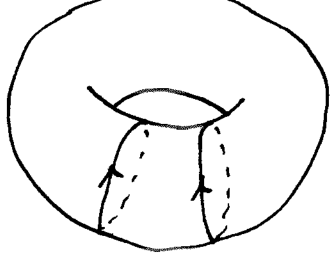


$C = \partial A$

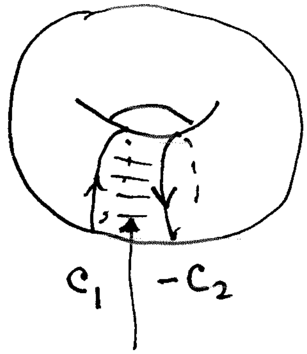


$C \neq \partial A$

This fact conveys topological information about the manifold. More precisely, introduce equivalence classes of cycles.



C_1 C_2



C_1 $-C_2$

A

We say, $C_1 \sim C_2$ if

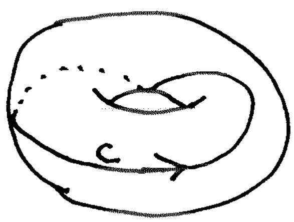
$$C_1 - C_2 = \partial A.$$

some A.

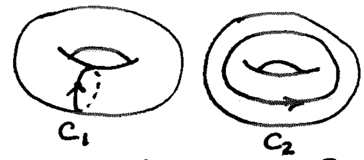
We say, C_1 and C_2 are <u>homologous</u>.

On $\mathbb{R}^3$, every 1-cycle is a boundary, $C_1 = \partial A_1$, $C_2 = \partial A_2$, so their difference is a boundary, too, $C_1 - C_2 = \partial(A_1 - A_2)$, and all 1-cycles belong to the same equivalence class. On $\mathbb{R}^3$, there is only one equivalence class. $[C]$, any 1-cycle C, in particular, $C = 0$ (the curve that doesn't go anywhere).

But on the 2-torus, two 1-cycles are equivalent iff their "winding numbers" n_1, n_2 are the same.



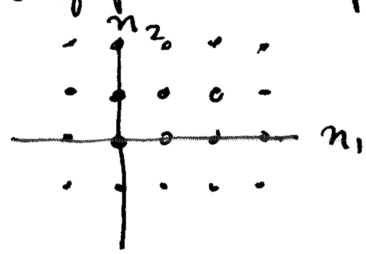
C has winding numbers $(1, 1)$.



$C = C_1 + C_2$.

Thus the equivalence classes of 1-cycles on T^2 are numbered by 2 integers (n_1, n_2). The space of equivalence classes is $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$.

Can think of this as lattice of points in a plane:



Can also think of it as an Abelian group, with composition law

$$(n_1, n_2)\,(m_1, m_2) = (n_1 + m_1,\ n_2 + m_2).$$

(vector addition of lattice vectors). This is expressed by saying,

$$H_1(T^2) = \mathbb{Z}^2$$

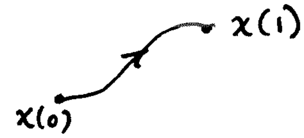
$$H_1(\mathbb{R}^3) = \{0\}.$$

H_1 (manifold) = 1st homology group of the manifold.

It is a topological invariant.

What does it mean to use expressions like $-C$, C_1+C_2, C_1-C_2, etc?

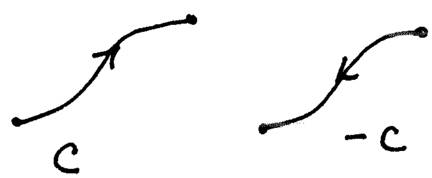
Let C = any oriented curve on a manifold M (not necessarily a cycle).



Formally, this is a map $C: [0,1] \to M : t \mapsto x(t)$. It is something you integrate over,

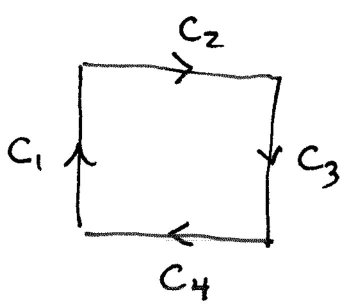
$$\int_C \alpha$$

where α = a differential 1-form (like $\vec{F}\cdot d\vec{x}$ on $\mathbb{R}^3$). Let $-C$ be the same curve traversed in the opposite direction,



So that $$\int_{-C} \alpha = -\int_C \alpha.$$

If you have several segments of a curve, (not necessarily concatenated) define their sum by sums of integrals:



$$\int_{C_1+C_2+C_3+C_4} \alpha = \int_{C_1} \alpha + \int_{C_2} \alpha + \int_{C_3} \alpha + \int_{C_4} \alpha.$$

Thus we can define "linear combinations" of curves with integer coefficients, and

$$\int_{\sum_i n_i C_i} \alpha = \sum_i n_i \int_{C_i} \alpha, \qquad n_i \in \mathbb{Z}.$$

These linear combinations of 1-dimensional, oriented curves with integer coefficients are called 1-chains. They are things you integrate 1-forms over.

Thus a 1-chain is a formal linear combination of oriented 1-curves with integer coefficients.

[Actually, in ~~homotopy~~ homology theory, one can choose the coefficients to be things other than integers. The favorite choices are $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{Z}_2$ (the spaces from which the coefficients are chosen). For now we use only $\mathbb{Z}$, but later we'll return to other types of coefficients.]

There is an huge infinite number of possible oriented curves on a given manifold. In our approach to ~~homotopy~~ homology theory, we will reduce this to a finite number by using a triangulation of a manifold (this is a homeomorphism between a polyhedron in $\mathbb{R}^n$ and the manifold M we wish to study).

The set of 1-chains that can be formed out of a finite # of N distinct oriented curves $\{C_1, \ldots, C_N\}$ is

$$\sum_{i=1}^{N} n_i C_i \, , \qquad n_i \in \mathbb{Z},$$

it is the space $\mathbb{Z}^N = \mathbb{Z} \times \ldots \times \mathbb{Z}$ consisting of N-vectors $(n_1, \ldots, n_N)$ with integer coefficients. This space is not a vector space (because $\mathbb{Z}$ is not a field), in fact it is usually regarded as an Abelian group in which the "multiplication law" is just addition of integer vectors and the identity is the zero vector $(0, \ldots, 0)$.

[Note: Nakahara writes $\mathbb{Z}\oplus\mathbb{Z}$ instead of $\mathbb{Z}\times\mathbb{Z} = \mathbb{Z}^2$. It is just the space of integer vectors (n_1, n_2), a lattice in the plane.].

Don't confuse $\mathbb{Z}^2$ with $\mathbb{Z}_2$; the latter is the set $\{0,1\}$ with add'n modulo 2.

$\mathbb{Z}^2$ = set $\{(n_1,n_2) \mid n_1, n_2 \in \mathbb{Z}\}$.

Begin with another excursion into group theory. We'll only need Abelian groups for homology theory, but for now we'll consider some issues that apply to any group (Abelian or non-Abelian).

Recall that a <u>group homomorphism</u> is a map $f: G \to X$ between groups G and X such that $f(g_1)f(g_2) = f(g_1 g_2)$, $\forall\, g_1, g_2 \in G$.

<u>Defs</u>: $\ker f = \{ g \in G \mid f(g) = e_X \}$ e_X = identity element in X
e_G = " " " G.

$\mathrm{im} f = \{ x \in X \mid x = f(g), \text{ some } g \in G\}$ (usual defn. of image).

Let $K = \ker f \subset G$
$I = \mathrm{im}\, f \subset X$. for brevity.

<u>Thm</u>: $\ker f = K$ is a normal subgroup of G.

<u>Proof</u>: ~~Let $k \in K$ and $g \in G$.~~ First show K is a subgroup.

Let $k_1, k_2 \in K$. Then $f(k_1)f(k_2) = f(k_1k_2)$

$$= e_X e_X = e_X,$$

so $k_1 k_2 \in K$.

Similarly show K satisfies other axioms of a group. Next show that K is a normal subgroup. This means that $gkg^{-1} \in K$ for all $g \in G$, $k \in K$. Easy:

$$f(gkg^{-1}) = f(g)f(k)f(g^{-1}) = f(g)\, e_X\, f(g^{-1})$$
$$= f(g)f(g^{-1}) = f(gg^{-1}) = f(e_G) = e_X.$$

Hence $gkg^{-1} \in K$.

Note: Should have pointed this out earlier, but if K is a normal subgroup of G, then the left cosets gK and right cosets Kg are identical (as subsets of G). This is what $gkg^{-1} \in K$ $\forall g \in G$, $k \in K$ means. If a subgroup is not normal, then the left and right cosets are generally different.

Thm: $\operatorname{im} f = I$ is isomorphic to the quotient group G/K,

$$\frac{G}{\ker f} \cong \operatorname{im} f. \qquad (\text{The isomorphism is } [g] \mapsto f(g).)$$

Proof: First show that there is a 1-2-1 corresp. between cosets in G/K and elements of I. To do this we need to show that 2 elements g_1, g_2 of G map onto the same element of I iff they belong to the same coset of G/K. So suppose $f(g_1) = f(g_2)$. This means $f(g_1)f(g_2)^{-1} = e_X = f(g_1 g_2^{-1}) \Rightarrow g_1 g_2^{-1} \in K \Rightarrow$ $g_1 = k g_2$ for some $k \Rightarrow g_1, g_2$ belong to same coset. Converse is easily proven. Thus $[g] \to f(g)$ is a bijection.
Next we need to show that $[g] \mapsto f(g)$ is an isomorphism. This is easy. (That is, it's a homomorphism, since we already know it's a bijection.)

Now specialize to Abelian groups. Note, for an Abelian group, every subgroup is normal, so the quotient group is always defined.

For Abelian groups, convenient to change notation, use "+" for "group multiplication" etc. Table:

general case		Abelian	
xy	$\longrightarrow$	$x+y$	
x^{-1}	$\longrightarrow$	$-x$	
e	$\longrightarrow$	0	
x^n	$\longrightarrow$	nx	
$\times$	$\longrightarrow$	$\oplus$	(Cartesian product)