

lectures

Chapter 2. Mathematical preliminaries. (Begin with terminology about functions)

maps, etc.

Psychology in physics is that a "function" is a real-valued function. Also, in physical applications we often speak of a "many-valued function".

In mathematics, however, a function, map, and mapping are all the same thing, a mapping between sets. If f maps set X into set Y , we write

$$f: X \rightarrow Y$$

↑ ↗
the domain the range.

$X,$
(Y need not be a set of
real numbers, can be any set).

The function is supposed to be defined for all $x \in X$ and is single-valued, i.e. $y = f(x)$ is a unique element of Y .

Here X, Y are sets and $x \in X, y \in Y$ are elements of those sets. Instead of writing $y = f(x)$ one often writes

$$f: x \mapsto y \quad [\text{means same as } y = f(x)].$$

Sometimes one writes

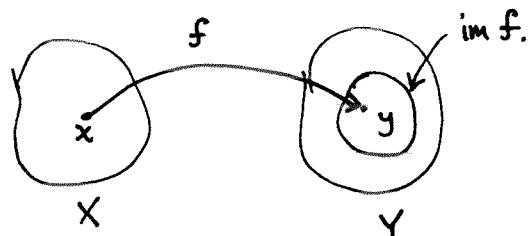
$$f: X \rightarrow Y: x \mapsto y \quad \left(\begin{array}{l} \text{gives info about domain, range,} \\ \text{and names elements } x \in X, y \in Y \\ \text{such that } y = f(x). \end{array} \right)$$

\checkmark (the domain)

Although f is defined for all $x \in X$, the y values obtainable as $y = f(x)$ for $x \in X$ do not have to fill up Y (the range).

Define image of f = $\text{im } f \subseteq Y$ by

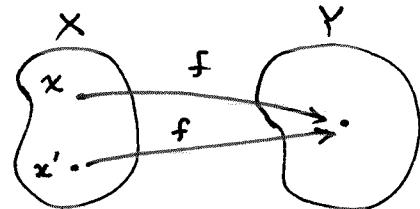
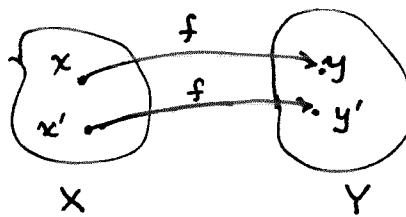
$$\text{im } f = \{ f(x) \mid x \in X \}.$$



In general, $\text{im } f$ is a (proper) subset of Y .

Classification of maps.

Injective or one-to-one means that distinct points of X are mapped to distinct points by f .



can't happen
with injective maps.

Injective: if $x \neq x'$, then $f(x) \neq f(x')$.

Surjective or onto means that $\text{im } f = Y$ (f fills up Y).

Bijection means one-to-one and onto.

The inverse function $f^{-1}: Y \rightarrow X$ exists iff f is bijection.

Sometimes we use the notation f^{-1} even when the inverse function doesn't exist. For example, we can interpret

$$f^{-1}(y) = \{x \in X \mid y = f(x)\}$$

this is a set of points
in X . = "inverse image of y "

or for $A \subset Y$, $f^{-1}(A) = \{x \in X \mid f(x) \in A\}$. "Inverse image of A ".

Other map stuff.

Constant map $c: X \rightarrow Y: x \mapsto c(x) = \text{indep. of } x$

Restriction of a map (that is, restriction to a subset of the domain).

Let $A \subset X$,

$$f|_A: A \rightarrow Y: x \mapsto f(x), \quad x \in A.$$

Notice that a bijection between two spaces places the points of those spaces in one-to-one correspondence.

Composition of maps:

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

$\underbrace{\hspace{1cm}}_{gof}$

$$gof: X \rightarrow Z: x \mapsto g(f(x))$$

(Note order of factors.)

(associative:
 $(h \circ g) \circ f = h \circ (g \circ f)$)

In the special case of a bijection,

$$f \circ f^{-1} = id_Y$$

$$f^{-1} \circ f = id_X$$

id = identity map.

$$id_X: X \rightarrow X$$

$$id_Y: Y \rightarrow Y$$

$$X \xrightarrow{f} Y$$

$\underbrace{\hspace{1cm}}_{f^{-1}}$

Inclusion Map: If $A \subseteq X$, define

$$f: A \rightarrow X: a \mapsto a, \quad a \in A.$$

often written $A \hookrightarrow X$.

We often have maps between spaces with structure (e.g., groups, vector spaces, ...). If f preserves this structure, then f is said to be a homomorphism. If f is also a bijection, then it is said to be an isomorphism.

We give 2 examples of spaces with structure: groups and vector spaces. (Digression to define these.)

Def. A group $\bullet G$ is a set of objects with a "multiplication rule" such that:

- 1) G is closed under multiplication
- 2) Mult. is associative, i.e. $(xy)z = x(yz), \forall x, y, z \in G$.
- 3) $\exists e \in G$ such that $ex = xe = x, \forall x \in G$.
- 4) $\forall x \in G, \exists x^{-1}$ s.t. $xx^{-1} = x^{-1}x = e$.

Here we are writing group multiplication by juxtaposing

- group elements, e.g. $x y$. In other contexts we might wish to indicate multiplication by some symbol, such as $x * y$ or $x + y$ etc.

The group axioms do not require the multiplication law to be commutative. If it is, we say the group is Abelian; otherwise, it is non-Abelian.

If f is a map between two groups G and H ,

$$f: G \rightarrow H$$

and if $f(a)f(b) = f(ab)$, $\forall a, b \in G$, then f is said to be a group homomorphism. If in addition f is a bijection, then f is said to be a group isomorphism. In this case f puts elements of G and H into one-to-one correspondence, preserving the multiplication law; in effect, it amounts to just relabeling the elements of G . If an isomorphism exists between groups G and H , then the groups are said to be isomorphic.

Now we define a vector space V over a field K . First, the most common fields used in this course are \mathbb{R} and \mathbb{C} , the real and complex numbers. A field is a set of objects with the usual operations of arithmetic, $+, -, *, \div$ defined, following the usual rules. For example, addition and multiplication are commutative and obey the usual distributive laws, you can divide by anything except 0 (and there exists a 0 and a 1) etc. The integers $\mathbb{Z} = \{\dots, -1, 0, 1, 2, \dots\}$ do not form

a field because you cannot always divide by integers.

For example, $\frac{3}{2}$ exists but is not an integer. Therefore you can't use \mathbb{Z} to make a vector space. The rationals \mathbb{Q} , however, do form a field. The quaternions \mathbb{H} ^{more later} do not form a field because the multiplication is not commutative.

The elements of the field K are called scalars. We call vector spaces over \mathbb{R} or \mathbb{C} real or complex vector spaces. It means the coefficients used in forming linear combinations are real or complex, it says nothing about the nature of the vectors themselves. For example, the space of Hermitian, 2×2 matrices is a real vector space, even though the matrices themselves contain complex numbers.

Elements of a vector space V can be added or multiplied by scalars,

$$\text{If } x, y \in V \text{ then } x+y \in V$$

$$\text{If } x \in V, \alpha \in K, \text{ then } \alpha x \in V$$

and they obey distributive laws,

$$\alpha(x+y) = \alpha x + \alpha y$$

etc.

If f is a map between vector spaces,

$$f: V \rightarrow W$$

such that

$$f(ax+by) = af(x)+bf(y), \quad \forall x, y \in V \\ \forall a, b \in K$$

then f is a ~~vector~~ vector space homomorphism. We usually just call it a linear map. The field K must be the same for both vector spaces. If bases are chosen in V, W , then f becomes represented by a matrix. If f is a bijection, then the linear map is invertible.

Now we come to the important concepts of equivalence relation, equivalence class, and quotient space, that will be used frequently in this course.

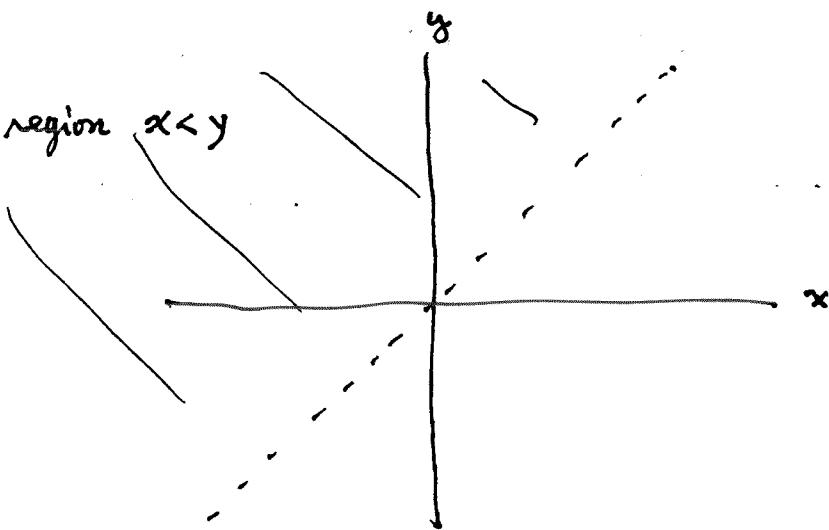
Idea: A relation is a thing like $=, \neq, >, <$ etc. (on real numbers)

Def: A relation R on a set X is a subset of $X \times X$,
and we write $x R y$ if $(x, y) \in R$. \uparrow Cartesian product.

The idea is that the relation is true if $(x, y) \in R$.

Example: Let $X = \mathbb{R}$, $R = <$
↑
real numbers ↑
the relation (don't confuse them).

Then $X \times X = \mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ = the plane,



$x < y$ if (x, y) lies above the diagonal line. Illustration of a standard trick in geometry: view something in a higher dimensional space.

Another point. Nakahara defines a relation R on set X as a subset of $X \times X$, i.e., the set of points (x, y) , $x, y \in X$, such that $x R y$ is true.

Another point of view is to define a relation as a map,

~~relation~~

$$R: X \times X \rightarrow \{\text{True, False}\}.$$

Then Nakahara's R is $R^{-1}(\text{True})$.

\uparrow
 Nakahara's \uparrow this R

Now, an equivalence relation is a relation that has all the formal properties of $=$.

Idea: Objects are equivalent if they are equal insofar as some of their properties are concerned (that is, if we ignore the other properties).

For example, suppose we consider two points in \mathbb{R}^3 to be equivalent if they have the same r coordinate (ignore θ, ϕ). This is an equivalence relation.

Def. A relation \sim on a set X is an equivalence relation if

- 1) $a \sim a$ (reflexive)
 - 2) $a \sim b \Rightarrow b \sim a$ (symmetry)
 - 3) $a \sim b$ and $b \sim c \Rightarrow a \sim c$ (transitive).
- $\left. \begin{array}{c} \\ \\ \end{array} \right\}$ for all
 $a, b, c \in X$.

Given an equivalence relation \sim on X , and $x \in X$, we define the

$$\text{equivalence class of } x = [x] = \{y \in X \mid y \sim x\}$$

(the set of all elements in X equivalent to x).

In the notation $[x]$, x is called the representative element of the equivalence class. Any other element equivalent to x can be chosen as the representative element, i.e. if $x \sim y$ then $[x] = [y]$.

Basic fact about equivalence relations is that they partition X into mutually disjoint subsets (the equivalence classes). That is, for all $x, y \in X$, either $[x] = [y]$ or $[x] \cap [y] = \emptyset$ (the empty set).

Proof: Assume $[x] \cap [y] \neq \emptyset$. Then there exists $a \in [x], [y]$ (a common element). This means $a \sim x$ and $a \sim y$, hence $x \sim y$, indeed, every element of $[x] \sim$ every element of $[y]$. Therefore $[x] = [y]$. But either $[x] \cap [y] = \emptyset$ or $[x] \cap [y] \neq \emptyset$. QED

Now define quotient space (important concept for this course).

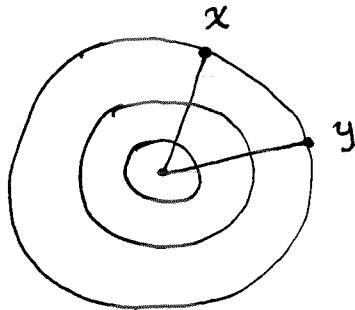
- Let \sim be an equiv. relation on X . Then
 $\xleftarrow{\text{notation for quotient space}}$

$$\text{the quotient space} = \frac{X}{\sim} = \{ \text{equiv. classes} \}.$$

In other words, each point of the quotient space corresponds to a whole equivalence class of points in X .

Examples.

- Ex. 1. Let $X = \mathbb{R}^3$ and let 2 points be equivalent if they have the same r coordinate. Then $x \sim y$ if x and y are same distance from origin, i.e. they lie on ~~a line~~ a sphere of a given radius. In fact, the equivalence classes are spheres.



$$[x] = [y] = \text{a sphere}$$

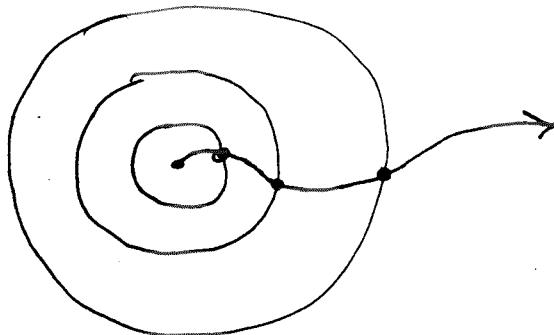
Obviously the spheres (plus the point at the origin) partition \mathbb{R}^3 into disjoint subsets. Since each sphere is characterized by its radius, the quotient space is the radial \mathbb{R}_+ -line,



Each point of this space corresponds to a single equiv. class.

Notice that the quotient space is not a subspace of the original space X , it is a new space.

On the other hand, it is possible to identify the quotient space with a subset of X , just not in a unique way. In the sphere example, consider a line that comes out from the origin and goes to ∞ . It need not be straight.



This line intersects each equivalence class (sphere) in one point, which can be taken as a representative element for that equiv. class. So the points of the line are placed in one-to-one correspondence with the equivalence classes, and the line itself can be identified as the quotient space.

The construction is not unique, however. The line can be pushed in any angular direction as long as it continues outward in the radial direction, and the new line will work just as well. That is, you can change the representative element in each equivalence class. This gives a simple picture of the

geometry that lies behind gauge transformations.

- Ex 2. Let $X = \mathbb{Z}$, let $n \sim m$ if $n-m = \text{even}$. Then compute some equiv. classes:

$$[0] = \{n \in \mathbb{Z} \mid n-0 = \text{even}\} = \{\text{even integers}\}$$

$$= \{\dots, -2, 0, 2, 4, \dots\},$$

$$[1] = \{n \in \mathbb{Z} \mid n-1 = \text{even}\} = \{\text{odd integers}\}$$

$$= \{\dots, -1, 1, 3, \dots\}.$$

These 2 equivalence classes fill up \mathbb{Z} , so

$$\frac{\mathbb{Z}}{\sim} = \{[0], [1]\} = \mathbb{Z}_2 \leftarrow \begin{matrix} \text{quot. space} \\ \text{notation for} \end{matrix}$$

$$= \text{"integers mod 2".}$$

Similarly we can define the "integers mod n " = \mathbb{Z}_n

$$= \{[0], [1], \dots, [n-1]\}.$$

- An interesting feature of this example is that the original set \mathbb{Z} has an addition law defined on it. Can this addition law be transferred or "projected" onto the quotient space to define an addition law there? That is, can we use our knowledge of how to add integers to define the addition of equivalence classes? The logical guess is that we should define $[n] + [m] = [n+m]$.

But we must show that this is meaningful. The left hand side labels two equivalence classes by their representative elements n and m , but any other representative elements $n' = n + 2a$ and $m' = m + 2b$, $a, b \in \mathbb{Z}$, would also work. We must show that the equivalence class on the right hand side is independent of the choices of representative elements on the left. In the present case,

$$[n'+m'] = [n+m + 2a+2b] = [n+m]$$

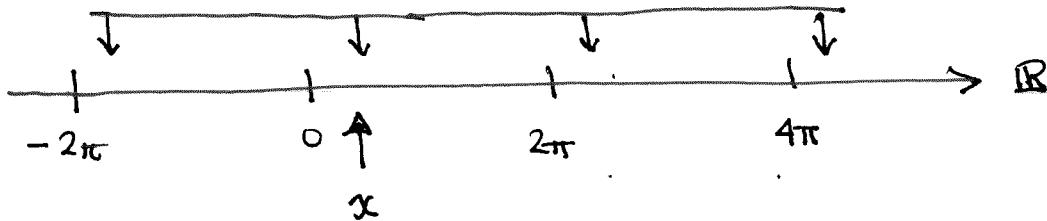
since $2a+2b$ is even. Thus the definition is meaningful.

The addition table is

	[0]	[1]
[0]	[0]	[1]
[1]	[1]	[0]

It is otherwise addition modulo 2.

Ex. 3. Like Ex 2 but we work with reals. Let $X = \mathbb{R}$ and let $x \sim y$ if $x - y = 2\pi n$, $n \in \mathbb{Z}$.

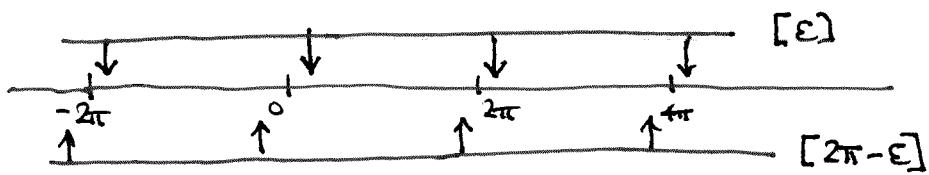


Then if x is a number, then $[x]$ is the set $\{x + 2\pi n \mid n \in \mathbb{Z}\}$.

- Then as x ranges over $0 \leq x < 2\pi$, the equivalence classes cover all of \mathbb{R} , so the quotient space can be identified with the $\frac{1}{2}$ -open interval $[0, 2\pi)$.

In this example we can define addition of quotient spaces just like we did in the previous example; it is now addition modulo 2π .

But in this example, the quotient space inherits something else from its parent besides an addition law: it inherits a topology. Topology is a way of specifying which points are near each other. Consider 2 points $x=\varepsilon$ and $x=2\pi-\varepsilon$ of the original space $X=\mathbb{R}$ when ε is small. These points are not close together. But the equivalence classes $[\varepsilon]$ and $[2\pi-\varepsilon]$ are close together, by any reasonable definition:



So in the quotient space $\Theta[0, 2\pi)$, as we approach 2π , we are actually approaching 0. Thus the $\frac{1}{2}$ -open interval should be closed to form a circle:

$$\frac{\mathbb{R}}{\sim} = \bigcirc = S^1 = \text{a circle.}$$

[Aside: S^n is standard notation for the " n -sphere", the set of points in Euclidean \mathbb{R}^{n+1} that are at a unit (or $\text{const} > 0$) distance from the origin. The n refers to the dimension of the surface of the sphere, not the embedding space. S^1 is the circle, S^2 the usual sphere in 3D space, etc.]

Ex. 4. Pure states in Quantum Mechanics are often loosely identified with ket vectors $|4\rangle$ in a Hilbert space \mathcal{H} . \mathcal{H} is a complex vector space. It is common to require $|4\rangle$ to be normalized, but for the following discussion we will assume only that $|4\rangle$ is normalizable, i.e., $|4\rangle \neq 0$. Thus, the space of ket vectors we will consider is $\mathcal{H} - \{0\}$. What is the relation between this space and the space of physical, pure states?

Every measurement process on a quantum system in a pure state produces a result that can be written

$$\langle A \rangle = \frac{\langle 4 | A | 4 \rangle}{\langle 4 | 4 \rangle},$$

where A is the operator that represents the measurement.

Thus no physical results change if $|4\rangle$ is replaced by $c|4\rangle$ where c is a nonzero complex number. Therefore a physical state does not correspond to a definite ket vector, but rather to an equivalence class of ket vectors in $\mathcal{H} - \{0\}$, where

$$|4\rangle \sim |\phi\rangle \quad \text{if } |4\rangle = c|\phi\rangle, \text{ some } c \in \mathbb{C}, c \neq 0.$$

The equivalence class ~~[4]~~ $[|4\rangle]$ is called a (complex) ray in \mathcal{H} . The usual ket vectors we use in QM $|4\rangle$ are actually representative elements of the ray (they are determined to within a normalization and a phase.)

Ex. 5. In the previous example, if \mathcal{H} is finite dimensional, e.g. for a spin system, then the quotient space is called a complex projective space. The general definition is the following. Let $X = \mathbb{C}^{n+1} - \{0\}$, and define

$$x \sim y \quad \text{if } x = cy \text{ where } c \in \mathbb{C} - \{0\}.$$

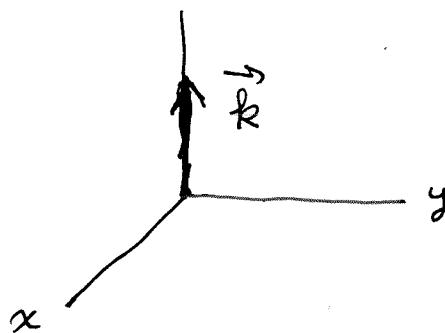
Then

$$\frac{\mathbb{C}^{n+1} - \{0\}}{\sim} = \mathbb{C}\mathbb{P}^n \quad (\text{complex projective space})$$

The $\mathbb{C}\mathbb{P}^n$ are "complex manifolds" (see ch. 8 of Nakahara).

- The n refers to the number of complex dimensions
 - (down by one from \mathbb{C}^{n+1} because we have removed
 the "normalization and phase".)

Ex. 6. In classical E+M, consider plane light waves
 of a given frequency ω propagating in the z -direction.
 What is the space of such light waves? Since $|\vec{k}| = \omega/c$,
 the magnitude of \vec{k} is known as well as its direction (\hat{z}),
 so the picture is



The most general plane wave solution of the vacuum Maxwell
 eqns with these constraints is

$$\vec{E}(\vec{x}, t) = \text{Re} [\vec{\alpha} e^{i(kz - \omega t)}]$$

where $\vec{\alpha}$ is an amplitude vector,

$$\vec{\alpha} = \alpha_x \hat{x} + \alpha_y \hat{y}.$$

- It must lie in the x-y plane because $\vec{E} \perp \vec{k}$. The
 components α_x and α_y of the amplitude are constants

(indep. of \vec{x}, t), and are allowed to be complex.

- Although \vec{E} is real, both the real and imaginary parts of a_x are physically meaningful (they can be measured), since they specify the amplitude and phase of E_x (for example, regarded as a fn. of t at fixed \vec{x}). Same for a_y . So,

$$a_x, a_y \in \mathbb{C},$$

$$\begin{pmatrix} a_x \\ a_y \end{pmatrix} \in \mathbb{C}^2 = \mathbb{R}^4$$

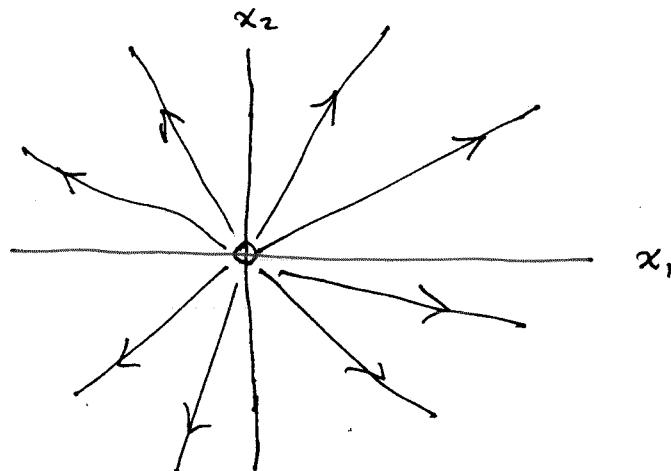
so the space of such plane light waves is \mathbb{C}^2 or \mathbb{R}^4 .

- Now consider the space of polarization states of these light waves. First, since the trivial solution $\vec{E}(\vec{x}, t) = 0$ doesn't have a polarization, we exclude it and consider the space $\mathbb{C}^2 - \{0\}$ or $\mathbb{R}^4 - \{0\}$ of solutions. Next, since polarization represents the direction of \vec{E} , which doesn't change if \vec{E} is scaled by any positive factor, we should form the quotient space (or "divide by" or "mod out by") the equivalence relation,

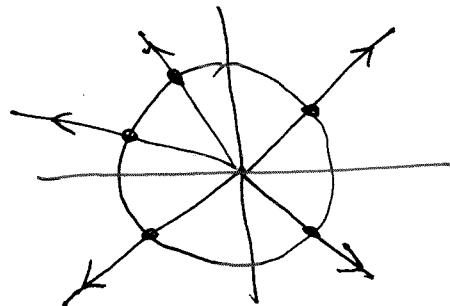
$$\vec{a} \sim \vec{a}' \quad \text{if} \quad \vec{a} = k \vec{a}', \quad k \in \mathbb{R}, \quad k > 0.$$

~~This~~ The equivalence classes of this \sim are one sided "rays"

in $\mathbb{P}^2 - \{\text{pt}\}$ or $\mathbb{R}^4 - \{\text{pt}\}$, for example, in $\mathbb{R}^2 - \{\text{pt}\}$ such rays would look like this:



In $\mathbb{R}^n - \{\text{pt}\}$ we obviously get one point (representative element) per ray if we intersect each ray with the sphere S^{n-1}



so the quotient space can be identified with such a sphere, in our case ($n=4$),

$$\frac{\mathbb{P}^2 - \{\text{pt}\}}{\sim} = \frac{\mathbb{R}^4 - \{\text{pt}\}}{\sim} = S^3.$$

obviously this is just adopting a "normalization convention", says,

$$|\vec{q}|^2 = |q_x|^2 + |q_y|^2 = E_0^2 = \underset{\text{const.}}{\overset{\text{some}}{=}}$$

This is still not the space of polarizations, however, because polarization measurements involve an averaging over many ~~cycles~~ cycles of the wave (a near necessity at optical frequencies). This means that the answers will not change if we shift the origin of time (we replace $t \rightarrow t + \frac{T}{2}$), which causes $\vec{a} \mapsto e^{-i\omega t} \vec{a}$. In other words, the polarization does not depend on the overall phase of the complex amplitude vector \vec{a} . (It does depend on the relative phases of a_x and a_y , however.) So now we should do another quotient operation. We let

$$\vec{a} \sim \vec{a}' \text{ if } \vec{a} = e^{i\phi} \vec{a}',$$

where \vec{a}, \vec{a}' are normalized, i.e. they belong to S^3 .

Now the equivalence classes are circles, obtained as ϕ goes from 0 to 2π . For all normalized vectors \vec{a} , these are nondegenerate circle, that is, they do not become points for any choice of \vec{a} . Thus S^3 can be divided or "foliated" into a family of circles. It is interesting that this cannot be done with S^2 (the circles of const. latitude become points at the N and S poles).

What is the quotient space? It turns out, topologically speaking, that it is the 2-sphere S^2 :

$$\frac{S^3}{\sim \text{(phase)}} = \frac{S^3}{S^1} = S^2$$

This is called the Hopf fibration, which represents S^3 as a circle bundle over S^2 . More on this later. It takes a little work (which we skip) to show that the quotient space is actually S^2 . The Hopf fibration is an example of a non-trivial fiber bundle; that means that it is not possible to choose a representative element from each equivalence class in a continuous manner inside S^3 . Thus the quotient space S^2 cannot be represented as a sub-manifold of S^3 , as we did above with the spheres.

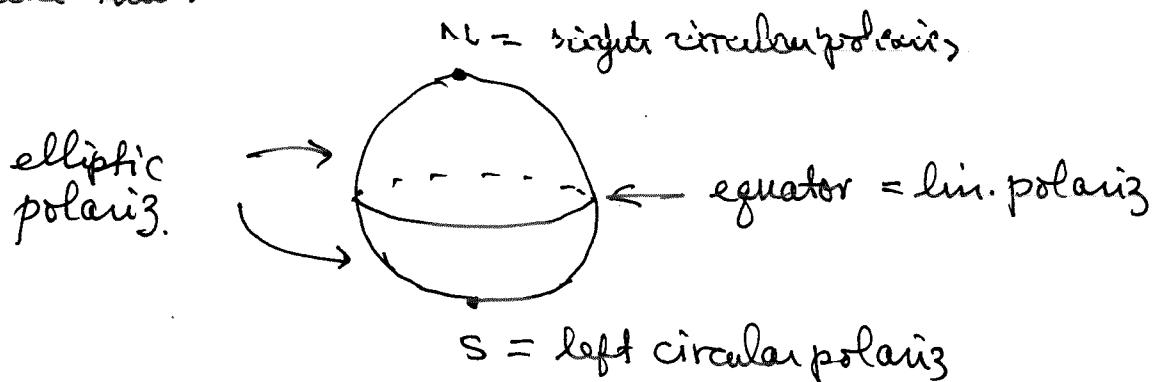
We have modded out first by a real and positive amplitude $a > 0$, and then a phase $e^{i\phi}$. Since every nonzero complex number can be written $a e^{i\phi}$, we could have obtained the same quotient space ^{in one step} by modding out by nonzero, complex multiplicative factors. This reproduces the definition of the complex projective space, so

$$\frac{\mathbb{C}^2 - \{0\}}{\sim \text{(complex mult)}} = \mathbb{C}\mathbb{P}^1.$$

But accepting the result above, we have

$$\mathbb{C}P^1 = S^2.$$

In optics, the sphere (called the Poincaré Sphere) looks like this:



This is the space of polarization states.

The mathematics of this light-wave-polarization example is the same as modding out by the normalization and phase of a 2-component spinor in QM. Call the spinor $|x\rangle$. The point on the 2-sphere is specified by a unit vector \hat{n} in \mathbb{R}^3 , and is given by

$$\hat{n} = \frac{\langle x | \vec{\sigma} | x \rangle}{\langle x | x \rangle},$$

where $\vec{\sigma}$ are the Pauli matrices. \hat{n} is the direction the spinor is "pointing in".

Ex. 7. The definition of $\mathbb{C}P^n$ can be modified to use real numbers, whereupon we obtain the "real projective space"

Some extra notes on lecture, covering some ideas that were introduced.

First, define the Cartesian product. If X and Y are sets, then the Cartesian product of X and Y is the set of all ordered pairs that can be constructed out of elements of these sets, that is,

$$X \times Y = \{ (x, y) \mid x \in X, y \in Y \}.$$

↑ ↓
 set "X" "times"
 big X small x

For example, $\mathbb{R} \times \mathbb{R}$ is $\{ (x, y) \mid x, y \in \mathbb{R} \} = \mathbb{R}^2$, and $\mathbb{C} \times \mathbb{C} = \mathbb{C}^2$ etc.

Next, the projection map. Let \sim be an equivalence relation on set X . Then every $x \in X$ belongs to one equivalence class, that is, $[x]$. Thus we define the projection map,

$$\pi: X \rightarrow \frac{X}{\sim}: x \mapsto [x]$$

It seems obligatory to use the symbol π for projection maps.