

Physics 250
Fall 2015
Homework 9
Due Friday, October 30, 2015

Reading Assignment: Nakahara, pp. 207-225. See also Frankel, pp. 80–109, 391–410.

Notes. We have skipped Nakahara’s section 5.5 (on the integration of differential forms). We’ll return to that next week.

On p. 208, Nakahara has an exercise to show that $O(1, 3)$ (the Lorentz group) has four connected components. This was mentioned in class (the four components are resolved by parity and time reversal). The identity component consists of the “proper” Lorentz transformations, the ones that do not reverse time or change the orientation of the spatial axes. This group (the identity component) has a double cover representing Lorentz transformations on spinors, which turns out to be $SL(2, \mathbb{C})$. The 4-component Dirac spinors are Lorentz transformed by a direct sum of two inequivalent representations of $SL(2, \mathbb{C})$.

On p. 210, Nakahara’s Eq. (5.112) is meaningless, at least in the center term, since there is no meaning to the bracket of vectors at a specific point $g \in G$ (unless $g = e$). This is just sloppy notation in proving the theorem, which is done right in the notes.

On p. 212, Nakahara’s definition of a 1-parameter subgroup is fine, it is a homomorphism $\phi : \mathbb{R} \rightarrow G$ (ϕ was denoted σ in the lectures). But in Eq. (5.118) he wants to define a vector field associated with the 1-parameter subgroup as the vector field that has the 1-parameter subgroup as an integral curve. The problem with this is that you cannot define a *field* by means of a single *integral curve*, which in general does not explore the whole manifold. You can use a single integral curve to define a vector at each point of the curve, but it doesn’t make a whole field. So at this point I stop reading up through Eq. (5.122). This subject is treated more carefully without such sloppy reasoning in the lecture notes.

On p. 213, the paragraph beginning with “Conversely, . . .” is covering the same territory covered in lecture, but I think it’s done more clearly in lecture. Nakahara’s notation $\sigma(t, g)$ means the same thing as $\Phi_t g$ in the lecture notes. When he writes,

$$\frac{d\sigma(t, g)}{dt} = X, \tag{9.1}$$

what he means is that X is the vector field whose advance map is σ (his notation) or Φ (mine). This is the general relation between vector fields and advance maps (on any manifold). I would write it this way,

$$\left. \frac{d}{dt} \right|_{t=0} \Phi_t^* = X, \tag{9.2}$$

where $X \in \mathfrak{X}(M)$, $\Phi_t : M \rightarrow M$ is the advance map, and both sides are understood to act on $\mathfrak{F}(M)$. Again, I think this material is covered more clearly in the notes.

On p. 216, Nakahara defines the action of a Lie group on a manifold. His map $\sigma : G \times M \rightarrow M$ is what I called Φ in lecture. Actually, I've tended to use the alternative notation, $\Phi(g, x) = \Phi_g x$, where $x \in M$, so that $\Phi_g : M \rightarrow M$. The remark at the top of p. 217 is a warning about an upcoming abuse of notation, in which $\Phi_g x$ is simply written gx .

On p. 224, Nakahara calls ad_a the "adjoint representation". I don't know anyone else who uses that terminology. I used the notation I_a for what he calls ad_a (it is the "inner automorphism" action). What he calls the "adjoint map" is what most people call the "adjoint representation."

1. (DTB) In class we developed the differential geometric theory of Lie algebras of a group by using left-invariant vector fields. But one can also define right-invariant vector fields. Do these give a different definition of the Lie algebra of a group (that is, of the Lie algebraic structure on $T_e G = \mathfrak{g}$)? In this problem we explore this question.

(a) A right-invariant vector field is defined by $X_V^R|_a = (R_{a*}|_e)V$, where the R superscript means "right," and $a \in G$, $V \in \mathfrak{g}$. When necessary to distinguish right- and left-invariant vector fields, we will write X_V^R and X_V^L , otherwise we will assume X_V (without a superscript) is left-invariant. Let $\sigma^R(t)$ be the integral curve of X_V^R passing through e at $t = 0$. Show that $\sigma^R(t) = \sigma^L(t)$, where $\sigma^L(t) = \sigma(t)$ is the integral curve of the left-invariant vector field X_V^L , passing through e at $t = 0$, which was discussed in lecture.

Hint: Do this by showing that $X_V^R = X_V^L$ when evaluated at a point on the integral curve $\sigma^L(t) = \sigma(t)$. The easiest way I found to do this was to represent a vector at a point by an equivalence class of curves $[c]$, and to note that the tangent map F_* can be defined by $F_*[c] = [F \circ c]$.

Thus, we can drop any superscript and just write $\sigma(t) = \exp(tV)$, as in the lecture notes. Find an expression for other integral curves of X_V^R (with other initial conditions) in terms of $\exp(tV)$.

(b) Suppose we define a new bracket $[\cdot, \cdot]^R$ of elements $V, W \in \mathfrak{g}$ by writing

$$[V, W]^R = [X_V^R, X_W^R]|_e. \tag{9.3}$$

What is the relation between this bracket and the bracket defined in class (which used left-invariant vector fields)?

Hint: Think about the geometrical meaning of the Lie bracket in terms of the commutativity of flows. By this time you should know how to compute flows for arbitrary initial conditions for both left- and right-invariant vector fields, in terms of $\exp(tV)$.

2. (DTB) Induced vector fields were discussed in class. We are given an action of a Lie group G on a manifold M by means of diffeomorphisms $\Phi_g : M \rightarrow M$. We let $V \in \mathfrak{g}$. We associate V with a vector field $V_M \in \mathfrak{X}(M)$ by writing,

$$V_M = \left. \frac{d}{dt} \right|_{t=0} \Phi_{\exp(tV)}^*, \tag{9.4}$$

where it is understood that both sides act on $\mathfrak{F}(M)$. More explicitly, this is

$$(V_M f)(x) = \left. \frac{d}{dt} \right|_{t=0} f(\Phi_{\exp(tV)} x), \quad (9.5)$$

for all $x \in M$, $f \in \mathfrak{F}(M)$.

Since we are using the symbol Φ for the action of G on M , let us use the symbol Ψ for advance maps. If X is a vector field (on any manifold), let the corresponding advance map be denoted by $\Psi_{X,t}$. The relation between the advance map and the vector field is

$$X = \left. \frac{d}{dt} \right|_{t=0} \Psi_{X,t}^*, \quad (9.6)$$

where again both sides are acting on scalar fields.

Returning to the induced vector field V_M , note that its integral curves are given by the action of the 1-parameter subgroup $\exp(tV)$ on an initial point, that is,

$$\Psi_{V_M,t} x = \Phi_{\exp(tV)} x. \quad (9.7)$$

(a) If $V, W \in \mathfrak{g}$, express the Lie bracket $[V_M, W_M]$ in terms of the Lie algebra bracket $[V, W]$. Hint: I found it useful to introduce the notation,

$$K_x : G \rightarrow M : g \rightarrow \Phi_g x, \quad (9.8)$$

and then to use K_x^* to pull back functions from M to G , where brackets can be evaluated. Note that $\text{im } K_x$ is the orbit of the group action through x .

(b) Find the induced vector fields for the actions $g \mapsto L_g$ and $g \mapsto R_{g^{-1}}$ of G on itself.

(c) The adjoint representation (my terminology, not Nakahara's) is the linear representation of G acting on its own Lie algebra, $g \mapsto \text{Ad}_g$, where $\text{Ad}_g : \mathfrak{g} \rightarrow \mathfrak{g}$ is a linear map defined by

$$\text{Ad}_g W = (I_{g*}|_e)W. \quad (9.9)$$

Here I_g is the inner automorphism, $I_g : G \rightarrow G : a \mapsto gag^{-1}$. The infinitesimal generator of this action is a vector field on \mathfrak{g} . However, since \mathfrak{g} is a vector space, the value of the vector field at any point $W \in \mathfrak{g}$ can be regarded as just another vector in \mathfrak{g} , parameterized by the point W at which the field is evaluated. That is, the vector can be translated parallel to itself to move its base to the origin. With this understanding, show that

$$\text{ad}_V W = \left. \frac{d}{dt} \right|_{t=0} \text{Ad}_{\exp(tV)} W = [V, W], \quad (9.10)$$

where $\text{ad}_V W$ is standard notation for this infinitesimal generator. As you see, it is an alternative notation for the bracket of elements of \mathfrak{g} .