Physics 250
Fall 2015
Homework 14
Due Friday, December 4, 2015

Reading Assignment: Lecture notes for Monday, November 23. That was the only lecture last week, but we covered the connection between Hodge star theory and cohomology, the Frobenius theorem, and the beginning of the geometry of variational principles, which is background before working on variational principles for general relativity. I'm only giving homework on the Frobenius theorem, but the material on Hodge star and cohomology is interesting so I recommend it. So is the material on the geometry of variational principles, but we're just beginning that so no homework for now.

1. (DTB) A problem on the Frobenius theorem. As discussed in class, a k-*distribution* over a manifold M ($\dim M = m$) is an assignment of k-dimensional subspaces ($k \le m$) in the tangent spaces T_xM for all $x \in M$ (or perhaps only over some region $U \in M$). The assignment is assumed to be smooth. We say that a k-distribution (in some region) is *integrable* if there exists an $(m-k)$-parameter family of k-dimensional surfaces in the region which are everywhere tangent to the distribution. The surfaces are only required to exist locally.

Let X, Y be vector fields on M which lie in a k-distribution Δ. If Δ is integrable, then the integral curves of both X and Y must lie in the surfaces tangent to Δ. Therefore if we follow the integral curves of X or Y in any order for any elapsed parameters, we must always remain on a given surface. Considering infinitesimal elapsed parameters, we see that the Lie bracket $[X, Y]$ must be tangent to the surface, that is, it must lie in the distribution. Thus, we have a theorem, that if Δ is integrable and $X, Y \in \Delta$, then $[X, Y] \in \Delta$. It turns out the converse is also true (at least locally): if for every vector fields $X, Y \in \Delta$, we have $[X, Y] \in \Delta$, then Δ is (locally) integrable.

Let θ^α, $\alpha = 1, \ldots, m-k$ be a set of linearly independent 1-forms that annihilate Δ. Show that the following four conditions are equivalent:

(i) For every $X, Y \in \Delta$, $[X, Y] \in \Delta$.

(ii) $d\theta^\alpha(X, Y) = 0$ for all vector fields $X, Y \in \Delta$.

(iii) There exist 1-forms ${\lambda^\alpha}_\beta$ such that $d\theta^\alpha = {\lambda^\alpha}_\beta \wedge \theta^\beta$.

(iv) $d\theta^\alpha \wedge \Omega = 0$, where $\Omega = \theta^1 \wedge \ldots \wedge \theta^{m-k}$.