

Physics 250
Fall 2015
Homework 13
Due Monday, November 30, 2015

Reading Assignment: Lecture notes for Monday, Nov. 16 and Friday, Nov. 19; Nakahara, 244–247; 250–251; 253–254; 261–282; 287–293. In Monday’s lecture I went quickly over or skipped a number of topics covered by Nakahara in Chap. 7 because they are usually covered in GR courses, which I assume you’ve either had or will have some day. Probably the most important fact from this set of topics is that given (M, g) , there is a unique connection that is metric-compatible ($\nabla g = 0$) and torsion-free ($T^\mu = 0$), namely, the Levi-Civita connection. This is the standard case used in conventional general relativity. I skipped some topics altogether, such as Riemann normal coordinates. On Friday I presented an introduction to the Hodge star, which I will conclude on Monday.

Notes. On pp. 258–259, Nakahara is trying to point out that if the connection coefficients $\Gamma_{\alpha\beta}^\mu$ vanish in some coordinate system, then it means that the rule of parallel transport (in those coordinates) is that the parallel transported vector has the same components as the original vector. This applies, for example, to parallel transport in a vector space, using linear coordinates. One can similarly define such a (trivial) connection on any parallelizable manifold, which includes group manifolds. Obviously, if the connection coefficients vanish (in some coordinates), then the curvature tensor $R^\mu{}_{\nu\alpha\beta}$ vanishes (in all coordinates).

On p. 253, Nakahara says that the variational condition gives the local extremum of the length of a curve between two points. Actually, it is only a stationary point, in general (a kind of a saddle point in function space).

On p. 265, Nakahara’s equation (7.68b) is wrong, but he never uses it in the subsequent derivation. Just below that, I can’t see what dividing by x' has to do with anything. Actually, the derivation of the geodesic equations are much easier if you just use a Lagrangian. In the present case, let

$$L(x, y, x', y') = \frac{1}{2} \frac{x'^2 + y'^2}{y^2}, \quad (13.1)$$

where the $1/2$ is only for convenience. (Also, we could have used the square root of the above expression, but the answers will be the same and the above expression is easier to work with.) You will find the Euler-Lagrange equations give you the geodesic equations immediately, and also (by Noether’s theorem on the ignorable coordinate x) they give you the integral (7.69).

On pp. 275–277, Nakahara derives the expression for the Weyl tensor, following the “elegant” coordinate-free approach of Nomizu. This derivation includes “straightforward but tedious” calculations. I found it straightforward and not so tedious just to do it in coordinates. Here are my main results. Write $\bar{g}_\mu = e^{2\sigma} g_{\mu\nu}$. Then you find,

$$\bar{\Gamma}_{\alpha\beta}^\mu = \Gamma_{\alpha\beta}^\mu + \delta_\alpha^\mu \sigma_{,\beta} + \delta_\beta^\mu \sigma_{,\alpha} - g_{\alpha\beta} g^{\mu\tau} \sigma_{,\tau}. \quad (13.2)$$

Then define

$$B_{\mu\nu} = \sigma_{;\mu\nu} - \sigma_{,\mu}\sigma_{,\nu} + \frac{1}{2}g_{\mu\nu}(g^{\alpha\beta}\sigma_{,\alpha}\sigma_{,\beta}). \quad (13.3)$$

Then calculate the Riemann tensor, and you find,

$$\bar{R}_{\mu\nu\alpha\beta} = e^{2\sigma}(R_{\mu\nu\alpha\beta} + g_{\mu\beta}B_{\nu\alpha} - g_{\mu\alpha}B_{\nu\beta} + g_{\alpha\nu}B_{\mu\beta} - g_{\beta\nu}B_{\mu\alpha}). \quad (13.4)$$

Taking traces, you find

$$\bar{R}_{\mu\nu} = R_{\mu\nu} - (m-2)B_{\mu\nu} - g_{\mu\nu}B^\alpha{}_\alpha, \quad (13.5)$$

and

$$\bar{R} = e^{-2\sigma}[R - 2(m-1)B^\alpha{}_\alpha], \quad (13.6)$$

where $m = \dim M$. Then substitute back, eliminate $B_{\mu\nu}$ in favor of $\bar{R}_{\mu\nu}$ and $R_{\mu\nu}$. This brings in $\text{tr } B = B^\alpha{}_\alpha$, and you eliminate that using Eqs. (13.5) and (13.6). You get an expression involving barred and unbarred tensors, which can be put into the form,

$$\bar{W}^\mu{}_{\nu\alpha\beta} = W^\mu{}_{\nu\alpha\beta}, \quad (13.7)$$

where

$$\begin{aligned} W_{\mu\nu\alpha\beta} &= R_{\mu\nu\alpha\beta} + \frac{1}{m-2}(g_{\mu\beta}R_{\alpha\nu} - g_{\mu\alpha}R_{\beta\nu} + g_{\alpha\nu}R_{\mu\beta} - g_{\beta\nu}R_{\mu\alpha}) \\ &+ \frac{1}{(m-1)(m-2)}(g_{\mu\alpha}g_{\beta\nu} - g_{\mu\beta}g_{\alpha\nu})R. \end{aligned} \quad (13.8)$$

I have not yet done the part of Chap. 7 on the derivation of the Einstein equations via a variational principle. I'm going to go deeper into the geometry of variational principles starting this week before going into that question.

In the handwritten notes I used a certain notation, and a different one in lecture, for the same thing. The two are

$$\text{sgn} \begin{pmatrix} \mu_1 & \cdots & \mu_r \\ \nu_1 & \cdots & \nu_r \end{pmatrix} = \delta_{\nu_1 \dots \nu_r}^{\mu_1 \dots \mu_r}. \quad (13.9)$$

1. (DTB) Consider the Poincaré half-plane, with metric

$$g = \frac{dx^2 + dy^2}{y^2}. \quad (13.10)$$

See pp. 265–266, and watch out for the error noted above. We only use the region $y > 0$. The geodesics in this metric are calculated in the book.

(a) Using the methods of Sec. 7.8.4, compute the curvature 2-form, $R^\alpha{}_\beta$. The Poincaré half-plane is a surface of constant negative curvature.

(b) Find the Killing vector fields for this metric. Do this any way you like, but I found them by writing the Killing vector field in the form,

$$X \frac{\partial}{\partial x} + Y \frac{\partial}{\partial y}, \quad (13.11)$$

and then finding a differential equation for X and Y .

(c) Show that the Killing vectors form a Lie algebra.

One can show that the advance maps generated by the Killing vectors for this problem can be expressed as fractional linear transformations,

$$z' = \frac{az + b}{cz + d}, \quad (13.12)$$

where $z = x + iy$. The identity component of the isometry group is $SO(2, 1)$.

2. (DTB) Let $A = A_\mu \theta^\mu$ be a 1-form, where $\theta^\mu = dx^\mu$ so we are working in a coordinate basis. It was shown in class that

$$d^\dagger A = -\frac{1}{\sqrt{|g|}} \left(\sqrt{|g|} A^\mu \right)_{;\mu}, \quad (13.13)$$

where indices are raised and lowered with $g_{\mu\nu}$. It was also mentioned that this is the same as

$$-A^\mu_{;\mu}, \quad (13.14)$$

where the covariant derivatives are computed with respect to the Levi-Civita connection. If you have taken a relativity course such formulas will be familiar; if not, they can be derived rather easily from the fact that

$$\Gamma^\mu_{\mu\nu} = \frac{\partial}{\partial x^\nu} \ln \sqrt{|g|}, \quad (13.15)$$

in the Levi-Civita connection.

(a) Let

$$B = \frac{1}{2} B_{\mu\nu} \theta^\mu \wedge \theta^\nu \quad (13.16)$$

be a 2-form on any space with a metric, where we continue writing $\theta^\mu = dx^\mu$. Find an expression similar to (13.13) for $d^\dagger B$. Find out how it is related to

$$B^{\mu\nu}_{;\nu}. \quad (13.17)$$

(b) Maxwell's equations in vacuum in curved spacetime, expressed in terms of the Hodge star, are $dF = 0$ and $d^\dagger F = 0$. Since $F = dA$, these imply $d^\dagger dA = 0$. Also, the condition for Lorentz gauge is $d^\dagger A = 0$, so in Lorentz gauge, the vector potential satisfies

$$\Delta A = 0, \quad (13.18)$$

where

$$\Delta = d^\dagger d + dd^\dagger. \quad (13.19)$$

Express ΔA in terms of covariant derivatives with respect to the Levi-Civita connection.