

Physics 250
Fall 2015
Homework 11
Due Friday, November 13, 2015

Reading Assignment: Monday's lecture (Nov. 2) was an introduction to symplectic geometry and Hamiltonian mechanics. See the lecture notes. There are topics covered in the book which were not covered in the lecture, but which are covered in the lecture notes. These include the behavior of cohomology groups under maps and the connection between deformation retracts and cohomology groups.

I discovered that 3 pages had been omitted from the lecture notes from last Friday (Oct 30). I have updated the notes.

At least one topic appears twice in the lecture notes, the subject of moving chains and a formula for

$$\frac{d}{dt} \int_{c_t} \omega, \quad (11.1)$$

where $c_t = \Phi_t \circ c_0$ where $\Phi_t = \exp(tX)$. That's because the newer version of the derivation of this formula is better than the old one, although the old one is more vivid geometrically.

Notes. Nakahara's proof of the Poincaré lemma (pp. 235–237) is straightforward to follow, but not very illuminating. A more geometrical approach to the subject was given in lecture. Notice that Nakahara's statement of Theorem 6.3 should say, "... any closed r -form on U for $r \geq 1$ is also exact." That is, the theorem as stated is not true for $r = 0$. Another way to state this version of the Poincaré lemma is to say that on a contractible region R , $H^r(R)$ is the same as $H^r(p_0)$, where p_0 is the contraction point.

For Poincaré duality (p. 238), just note the result, Eq. (6.37). It is impossible to understand the logic, based on material we've covered so far.

Nakahara's theorem 6.5, p. 241, is trivial, if you note that $\pi_1(M) = \{0\}$ implies $H_1(M) = \{0\}$ implies $H^1(M) = \{0\}$.

1. A *symplectic form* on a manifold P is a 2-form $\omega \in \Omega^2(P)$ that is closed, $d\omega = 0$ and nondegenerate, that is

$$\det \omega_{\mu\nu} = \omega\left(\frac{\partial}{\partial z^\mu}, \frac{\partial}{\partial z^\nu}\right) \neq 0, \quad (11.2)$$

in all coordinates z^μ on P . Because the rank of an antisymmetric matrix is always even, $\dim P = 2n$ is an even number. A manifold P endowed with a symplectic form is called a *symplectic manifold*. Although every cotangent bundle T^*M is a symplectic manifold, there are many symplectic manifolds, important in mechanics, that are not cotangent bundles.

Although on the cotangent bundle $P = T^*M$ canonical coordinates $z^\mu = (q^i, p_i)$ can be used, in this problem we will let z^μ be arbitrary coordinates.

The symplectic form ω at a point $z \in P$ can be regarded as a linear map,

$$\omega|_z : T_z P \rightarrow T_z^* P : X \mapsto -i_X \omega, \quad (11.3)$$

which in coordinates is given by

$$X^\mu \rightarrow \omega_{\mu\nu} X^\nu, \quad (11.4)$$

And since ω is nondegenerate, the inverse map exists. Define a tensor $J^{\mu\nu}$ by

$$\omega_{\mu\sigma} J^{\sigma\nu} = \delta_\mu^\nu, \quad (11.5)$$

so that J at a point $z \in P$ can be regarded as a map,

$$J|_z : T_z^* P \rightarrow T_z P. \quad (11.6)$$

In components, the map is

$$\alpha_\mu \rightarrow J^{\mu\nu} \alpha_\nu. \quad (11.7)$$

These roles of ω and J (converting vectors to covectors and vice versa) are entirely analogous to the “raising and lowering” of indices with the use of a metric tensor. The symplectic manifold P , however, does not have a metric tensor.

Let $A \in \mathfrak{F}(P)$ be a function on P , and associate A with a vector field X_A by

$$(X_A)^\mu = J^{\mu\nu} \frac{\partial A}{\partial z^\nu}. \quad (11.8)$$

We call A a “Hamiltonian function” and X_A a *Hamiltonian vector field*. The tensor J is also used to define the Poisson bracket; if $A, B \in \mathfrak{F}(P)$, then

$$\{A, B\} = \frac{\partial A}{\partial z^\mu} J^{\mu\nu} \frac{\partial B}{\partial z^\nu} = J(dA, dB). \quad (11.9)$$

(a) Show that the Jacobi identity,

$$\{A, \{B, C\}\} + \{B, \{C, A\}\} + \{C, \{A, B\}\} = 0 \quad (11.10)$$

is equivalent to the condition $d\omega = 0$.

(b) Show that

$$[X_A, X_B] = -X_{\{A, B\}}. \quad (11.11)$$

2. In class we derived the formula,

$$\int_0^T dt \int_c \Phi_t^* i_X d\omega = \int_c \Phi_T^* \omega - \int_c \omega - \int_0^T dt \int_c d\Phi_t^* i_X \omega, \quad (11.12)$$

where c is an r -chain on M , ω is an r -form on M , X is a vector field on M and Φ_t is the associated advance map. Let $m = \dim M$. On a region of M diffeomorphic to the m -ball (the region $r < 1$ in \mathbb{R}^m), let X be the flow that contracts the ball to its center,

$$X^\mu(x) = -x^\mu. \tag{11.13}$$

Use this flow to derive the Volterra formula for the potential of a closed form ω .

3. (DTB) In lecture on Friday it was explained what a connection is (it is a way of identifying nearby tangent spaces), but we did not give any examples of how a connection may be defined. We will remedy this situation as time goes on, but this is a beginning.

Let M be a submanifold of Euclidean \mathbb{R}^n , with $\dim M = m \leq n$. The metric on \mathbb{R}^n is

$$g = \sum_{i=1}^n dx^i \otimes dx^i, \tag{11.13}$$

where $\{x^i, i = 1, \dots, n\}$ are the standard coordinates on \mathbb{R}^n . Let $\{x^\mu, \mu = 1, \dots, m\}$ be coordinates on M , which is specified by functions $x^i = x^i(x^\mu)$. Let the metric on the submanifold be the metric on \mathbb{R}^n , restricted to the submanifold. The metric on M has components $g_{\mu\nu}$. Let x^μ and $x^\mu + \xi^\mu$ be coordinates of two nearby points (call them x and $x + \xi$) on M (ξ^μ is infinitesimal). Now define a connection on M as follows. We take a tangent vector X in $T_x M$, reinterpret it as a tangent vector in $T_x \mathbb{R}^n$, parallel transport it over to $T_{x+\xi} \mathbb{R}^n$ by using the vector space structure of \mathbb{R}^n , then project it onto $T_{x+\xi} M$ using the metric in \mathbb{R}^n . Find the connection coefficients $\Gamma_{\alpha\beta}^\mu$ in terms of $g_{\mu\nu}$.