Physics 250
Fall 2015
Homework 10
Due Friday, November 6, 2015

Reading Assignment: Nakahara briefly considers the symplectic geometry of classical mechanics on pp. 202–203. If you read this section, note that Nakahara's Eq. (5.89) should read, $\theta = p_\mu \, dq^\mu$. We will go into this subject more deeply, in pieces. The first piece was the discussion of Noether's theorem, an application of induced vector fields, which is in the lecture notes.

For the other material covered this week, please read, in addition to the lecture notes, Nakahara pp. 205–206, and pp. 226–235. We will take up the Poincaré lemma and potentials next week.

By the way, I have added some text to Notes 2 on the course web site, so if you downloaded it earlier you may not have the latest version. I may continue to add a bit more text to those notes, so they may not be in final form even yet.

Notes. On p. 204, Nakahara defines a "volume form" as an m-form on an m-dimensional manifold that vanishes nowhere, and he discusses the theorem that such a form exists iff the manifold is orientable. I'd prefer it if he did not call this a "volume form," since I'm afraid that word suggests the usual volume computed with a metric. There is no metric in Nakahara's construction. If the manifold does have a metric, then the volume is defined in a unique way (whereas Nakahara's "volume forms" are far from unique). Nakahara uses his "volume forms" in his development of the subject of integration. I have done it a different way in lecture.

On p. 206, Nakahara introduces the "partition of unity," a useful concept theoretically, but I have skipped it in lecture because we can get by without it. See the discussion in Frankel.

On p. 208, Nakahara has an exercise to show that $O(1,3)$ (the Lorentz group) has four connected components. This was mentioned in class (the four components are resolved by parity and time reversal). The identity component consists of the "proper" Lorentz transformations, the ones that do not reverse time or change the orientation of the spatial axes. This group (the identity component) has a double cover representing Lorentz transformations on spinors, which turns out to be $SL(2,\mathbb{C})$. The 4-component Dirac spinors are Lorentz transformed by a direct sum of two inequivalent representations of $SL(2,\mathbb{C})$.

Nakahara's approach to integration uses singular simplexes, whereas I used singular cubes in lecture. Since a cube can be divided into simplexes or mapped into a simplex, it makes no difference for the development of the theory which is used. I chose cubes because it makes the proof of Stokes' theorem a little easier. In general, you may think of a chain as a linear combination of "singular" objects, that is, mappings from some standard region in $\mathbb{R}^r$ to M, which do not have to be injective or of maximal rank.

On p. 233, Nakahara says that d is the adjoint of ∂. This is incorrect, according to his definition of the adjoint, which requires a metric. It is correct to say, however, that d is the pull-back of ∂. More exactly, $d_r : \Omega^r(M) \to \Omega^{r+1}(M)$ is the pull-back of $\partial_{r+1} : C_{r+1}(M) \to C_r(M)$, as looks reasonable if we identify $C_r(M)^*$ with $\Omega^r(M)$. The pull-back does not require a metric.

In Corollary 6.1 on p. 234, Nakahara means to say that the set of cohomology classes $\{[c_i]\}$ are linearly independent (just $[c_i] \neq [c_j]$ is not enough).

1. The configuration space of a rigid body with one point fixed is $SO(3)$. To establish a one-to-one correspondence between configurations in a physical sense and matrices $A \in SO(3)$ we proceed as follows. Let the fixed point of the rigid body be the origin of a set of right-handed, inertial coordinates. Let $\{\hat{\mathbf{e}}_i, i = 1, 2, 3\}$ be the unit vectors of this coordinate system. We call $\{\hat{\mathbf{e}}_i\}$ the *space frame*. Let $R : \mathbb{R}^3 \to \mathbb{R}^3$ be a proper rotation leaving the origin fixed. We associate R with a matrix $A \in SO(3)$ by writing,

$$A_{ij} = \hat{\mathbf{e}}_i \cdot (R\hat{\mathbf{e}}_j). \tag{10.1}$$

Choose a reference orientation for the rigid body, and let the mass density be $\rho(\mathbf{x})$ in the reference orientation. The reference orientation can be chosen arbitrarily; it need not be an initial condition. The actual orientation at some time t is specified by a matrix $A(t) \in SO(3)$, which rotates the reference orientation to the actual orientation. Thus, the orbit of the classical system is a trajectory $A(t)$ through $SO(3)$ that satisfies the equations of motion.

Define time-dependent unit vectors by

$$\hat{\mathbf{f}}_i(t) = R(t)\hat{\mathbf{e}}_i. \tag{10.2}$$

The vectors $\{\hat{\mathbf{f}}_i\}$ constitute the *body frame*. You can think of $\{\hat{\mathbf{e}}_i\}$ as being imbedded in the body when the body is in its reference orientation, and $\{\hat{\mathbf{f}}_i\}$ as being fixed in the body in its actual orientation, rotating with the body.

The moment of inertia tensor in the body frame is the same as the moment of inertia tensor in the space frame when the body is in the reference orientation. It is given by

$$M_{ij} = \int d^3\mathbf{x}\, \rho(\mathbf{x})(|\mathbf{x}|^2\delta_{ij} - x_i x_j). \tag{10.3}$$

The moment-of-inertia tensor has components which are time-independent in the body frame, but the space frame components are time-dependent.

The angular velocity $\boldsymbol{\omega}$ is defined by

$$\mathbf{v} = \boldsymbol{\omega}\times\mathbf{x}, \tag{10.4}$$

where $\mathbf{x}$ is the position of a particle fixed in the body, and $\mathbf{v}$ is its velocity. This equation can be referred either to the body frame or the space frame.

The kinetic energy of the rigid body is

$$T = \frac{1}{2}\boldsymbol{\omega} \cdot M \cdot \boldsymbol{\omega}, \tag{10.5}$$

where $\boldsymbol{\omega}$ is the angular velocity and where it is understood that both $\boldsymbol{\omega}$ and M are taken with respect to the same frame (space or body).

(a) Nakahara's matrices X_x, X_y, X_z on the bottom of p. 223 span the Lie algebra of $SO(3)$. For this problem, call these matrices $\{J_i, i = 1, 2, 3\}$, but remember that the subscript on J_i is a label of the matrix, not a component index. The actual components of these matrices are

$$(J_i)_{jk} = -\epsilon_{ijk}. \tag{10.6}$$

Recall that in class, a tangent vector X to the group manifold of a real matrix group was associated with a matrix X (same symbol) by writing,

$$X = \sum_{ij} X_{ij} \frac{\partial}{\partial x_{ij}}, \tag{10.7}$$

where $\{x_{ij}\}$ are the components of the matrix (hence coordinates on matrix space). For example, if evaluated at the identity, this matrix represents an element of the Lie algebra. For practice you should work out the matrix representing a left-invariant (or right-invariant) vector field, evaluated at an arbitrary point (matrix) of the group.

For the rigid body problem, the velocity of the system is specified by a matrix $\dot{A}$. Express this as a linear combination of the left-invariant vector fields, using the basis $\{J_i\}$ of the Lie algebra, and interpret the components in ordinary (non-differential geometric) language. Do the same for the right-invariant vector fields.

(b) For two dimensional rotations, we can write

$$\omega = \frac{d\theta}{dt}, \tag{10.8}$$

where $\theta(t)$ is the angle of rotation. In three dimensions, we can write the angular velocity in terms of the Euler angles (θ, ϕ, ψ) and their derivatives. The actual formulas for the space components of the angular velocity are

$$\begin{aligned} \omega_x &= -\sin\phi\,\dot{\phi} + \cos\phi\sin\theta\,\dot{\psi}, \\ \omega_y &= \cos\phi\,\dot{\theta} + \sin\phi\sin\theta\,\dot{\psi}, \\ \omega_z &= \dot{\phi} + \cos\theta\,\dot{\psi}, \end{aligned} \tag{10.9}$$

but you do not need to do anything with these equations except to note that they are complicated and not very symmetrical. Nevertheless, compared to the two-dimensional case, the question arises whether these equations are messy because of the definitions of the Euler angles. Maybe with some other definition of angles, call them $\boldsymbol{\theta} = (\theta_x, \theta_y, \theta_z)$, we would have an obvious generalization of the two-dimensional formula,

$$\boldsymbol{\omega} = \frac{d\boldsymbol{\theta}}{dt}. \tag{10.10}$$

Can such angles $\boldsymbol{\theta}$ be found? Explain why or why not.

(c) In a recent homework, you worked out the Euler-Lagrange equations for a mechanical system when the velocity is expressed in a non-coordinate basis. You should have obtained,

$$\frac{d\pi_\mu}{dt} = e_\mu^{\ \nu} \frac{\partial \bar{L}}{\partial x^\nu} - c^\sigma_{\mu\nu}\, v^\nu \pi_\sigma, \tag{10.11}$$

(See the earlier homework for the notation).

For the force-free rigid body, the Lagrangian is just the kinetic energy. Use Eq. (10.11) to obtain the equations of motion for the force-free rigid body. Do this in the body frame.

2. It was explained in class that if M is the configuration space of a mechanical system, then the Lagrangian is a scalar field on TM, the tangent bundle, that is

$$L : TM \to \mathbb{R}. \tag{10.12}$$

Using x^i as coordinates on M (not necessarily rectangular coordinates) and $(x^i, \dot{x}^i)$ as coordinates on TM, we can say in coordinate language that $L = L(x^i, \dot{x}^i)$. Sometimes it is more clear to write v^i instead of $\dot{x}^i$.

The definition of the canonical momentum in mechanics is

$$p_i = \frac{\partial L}{\partial \dot{x}^i}, \tag{10.13}$$

which, as explained in class, we view geometrically as a map from TM (the tangent bundle) to T^*M (the cotangent bundle). That is, we regard (x^i, p_i) as coordinates on T^*M, in which a 1-form at a point $x \in M$ is written $p_i\, dx^i|_x$. Let us write this map as

$$F : TM \to T^*M. \tag{10.14}$$

We will call F the *Legendre map* (when viewed as a coordinate transformation, as is commonly done in books on mechanics, it is called the *Legendre transformation*). Notice that F maps the fiber T_xM of the tangent bundle into the fiber T^*_xM of the cotangent bundle, that is, at the same point x. So the coordinates x^i don't change under the Legendre map.

If the condition

$$\det \frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j} \neq 0 \tag{10.15}$$

is met everywhere on TM, then the map F has full rank everywhere and is a diffeomorphism. In this case we say that the Lagrangian is *regular*. Regular Lagrangians are normal in nonrelativistic mechanics, but irregular Lagrangians are normal in relativistic mechanics and field theory.

The Hamiltonian is usually defined as

$$H = p_i\, \dot{x}^i - L, \tag{10.16}$$

where p_i is given by Eq. (10.13). By this definition, H emerges as a function of x^i and $\dot{x}^i$, that is, as a function on TM. However, for regular Lagrangians, it is customary to eliminate the $\dot{x}^i$ in favor of the p_i to express H as a function of (x^i, p_i). Geometrically this amounts to defining H as a scalar field on T^*M by pushing it forward by the Legendre map F, that is, using the pull-back of F^{-1}. Then we think of H as a function,

$$H : T^*M \to \mathbb{R}. \tag{10.17}$$

Then one can show that the Euler-Lagrange equations coming from the Lagrangian L are equivalent to Hamilton's equations,

$$\dot{x}^i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial x^i}. \tag{10.18}$$

Suppose a Lagrangian is irregular. The rank of a map is defined as the rank of its Jacobian or tangent map, which in general is a function of position. Suppose the rank of F is constant (the same everywhere on TM) but not maximal.

Let $n = \dim M$. In the regular case, the rank of F is maximal, that is $2n = \dim TM = \dim T^*M$. So let us look at the case where $\operatorname{rank} F < 2n$.

It is convenient to restrict the domain of F to a single fiber of TM, since that fiber is mapped into the corresponding fiber of T^*M. Let us write,

$$F|_x : T_xM \to T_x^*M : (x, v) \to F(x, v) = (x, p), \tag{10.19}$$

in a notation that is hopefully obvious. Then if F has rank $2n - s$ at $x \in M$, then $F|_x$ has rank $n - s$, since the x coordinates are not changed by F.

Let us define the Hamiltonian, as a function on TM, by the same formula (10.16) that we use in the regular case. But if F does not have maximal rank, then we cannot push H over to become a function on T^*M. The best we can hope for is to define H as a function on the image of F, a subset of T^*M. In nice cases this will be a submanifold of T^*M, and it is called the "primary constraint manifold" in the language of Dirac constraints. Call this manifold C. But if the map $F : TM \to C$ is not injective, then we still cannot use F to push H forward into a function on C, because if two different points of TM map to the same point of C, then we wouldn't know which value of the two values of H to assign to the point of C. Unless it should happen that H has a constant value on the preimage of a point $z \in C$, then we could push H forward into a function on C.

(a) Let $C = \operatorname{img} F = F(TM) \subset T^*M$, and let $z \in C$. Suppose the preimage $F^{-1}z$ is a connected manifold (a submanifold of TM). Show that $H : TM \to \mathbb{R}$ is constant on this preimage. Thus, this constant value of H on the primage can be assigned as the value of H on $z \in C$, and we obtain a function $H : C \to \mathbb{R}$.

Hint: Do this by showing that $H : TM \to \mathbb{R}$ is constant along a vector X tangent to the preimage of $z \in C$. Note that such a vector X lies in the kernel of the tangent map F_*.

(b) Consider a covariant formulation of Lagrangian and Hamiltonian mechanics for a free particle in special relativity. The same geometry occurs for more sophisticated problems, such as a particle in the presence of an electromagnetic and/or gravitational field in special or general relativity, but this simple example is enough to see the geometry.

Let the configuration space be space-time, with coordinates $x^\mu = (t, x, y, z)$ (and with $c = 1$). Since t is a coordinate and not a parameter of the orbits, let's use an arbitrary parameter for the orbits λ and write $dx^\mu/d\lambda$ instead of $\dot{x}^\mu$. We can also write $v^\mu = dx^\mu/d\lambda$. Then $M = \mathbb{R}^4$ with coordinates x^μ and $TM = \mathbb{R}^8$ with coordinates (x^μ, v^μ). Also, $T^*M = \mathbb{R}^8$ with coordinates (x^μ, p_μ). Let the Lagrangian be

$$L = m\Big(g_{\mu\nu}\,\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\Big)^{1/2}, \tag{10.20}$$

where $g_{\mu\nu} = \text{diag}(+1, -1, -1, -1)$. Note that the proper time is

$$d\tau^2 = g_{\mu\nu}\,dx^\mu dx^\nu, \tag{10.21}$$

but λ is an arbitrary parameter, not necessarily proper time.

Define the momentum by

$$p_\mu = \frac{\partial L}{\partial v^\mu}, \tag{10.22}$$

which defines the Legendre map : $TM \to T^*M$.

Find the rank of the Legendre map restricted to a single fiber, as in (10.19). Find $\text{img}\,F$ as a submanifold of T^*M, that is, describe the primary constraint manifold. For $z \in C$, find $F^{-1}z$. If more than one point of TM maps into one point of T^*M, we might worry that we lose some physical information by going over to a Hamiltonian description. Show that in this case, all points on the preimage of some $z \in C$ are the same physically. Find the Hamiltonian as a function on C.