Physics 250
Fall 2015
Homework and Notes 1
Due Friday, September 4, 2015

About Homework: I will try to have weekly homework, but there may be some weeks without it. Nakahara's problems are usually not very good, so I will try to do better. The homework will be made available on the web site by Friday of each week, and will be due at 5pm on Friday of the following week, in the envelope hanging outside my office (449 Birge).

In any homework exercise marked "DTB" (meaning, "Done This Before"), you may get full credit by simply stating, "DTB". Please also say where you have done it before, such as, "Math 799 at Appalachian State Teacher's College" or "Self Study".

Reading Assignment: Lecture notes for 8/27/15, Nakahara, Chapter 1. Also please read into Chapter 2 of Nakahara to cover approximately the same material as in the lecture notes.

1. (DTB) This problem is an exercise in equivalence classes and quotient spaces.

The *complex projective space* $\mathbb{C}P^n$ is an important space. It was introduced in class as the space of complex rays in a Hilbert space. A ray consists of nonzero wave functions that are related to one another by multiplication by a nonzero complex number. Since the complex number can be broken into a real and positive amplitude and a phase, we can say that two wave functions are equivalent if they are related by a real and positive multiplicative factor (which only changes the normalization), and by an overall phase. If the Hilbert space is finite dimensional, then the n in $\mathbb{C}P^n$ is finite. This equivalence class is important physically because any two wave functions belonging to one of these equivalence classes are equivalent physically (they produce the same probabilities for all physical measurements that can be carried out on the system).

We define two nonzero points of $\mathbb{C}^{n+1}$ to be equivalent, $x \sim y$ for $x, y \in \mathbb{C}^{n+1}$, if $x = cy$ for some nonzero complex number c. Then the complex projective space $\mathbb{C}P^n$ is defined as

$$\mathbb{C}P^n = \frac{\mathbb{C}^{n+1} - \{0\}}{\sim}. \tag{1}$$

Notation like this is fine if we are clear on what equivalence relation $\sim$ means, but in this case if we want greater clarity we can write

$$\mathbb{C}P^n = \frac{\mathbb{C}^{n+1} - \{0\}}{\mathbb{C} - \{0\}}. \tag{2}$$

The space of spin wave functions for a spin-$\frac{1}{2}$ particles is $\mathbb{C}^2$, that is, it is a 2-component spinor. Let us write

$$\psi = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \tag{3}$$

for this spinor, where $z_1, z_2 \in \mathbb{C}$ and $\psi \in \mathbb{C}^2$. We shall carrying out the quotient operation in (1) or (2) in two steps, first dividing by the equivalence relation that takes care of the normalization, and then the one that takes care of the phase.

In the first step, we define $\sim$ on $\mathbb{C}^2 - \{0\}$ by saying two nonzero spinors ψ and ϕ are equivalent if $\psi = a\phi$, for a real and $a > 0$, that is, $a \in \mathbb{R}_+$. It was explained in class that the equivalence classes for this equivalence relation are rays in $\mathbb{R}^4$, that is, half-lines coming out of the origin of $\mathbb{R}^4$ but not including the origin itself. It was also explained that each ray can be labeled by a representative element ψ such that $|\psi|^2 = \langle\psi|\psi\rangle = 1$. Such representative elements lie on the unit sphere in $\mathbb{R}^4$, so that

$$\frac{\mathbb{C}^2 - \{0\}}{\mathbb{R}_+} = S^3. \tag{4}$$

Notice that in this case the quotient space (S^3) can be regarded as a subset of the original space ($\mathbb{C}^2 - \{0\}$), although as remarked in class it is usually best to think of a quotient space (the space of equivalence classes) as a different space from its parent.

In the next step we define two normalized spinors to be equivalent, $\psi \sim \phi$, for $\psi, \phi \in S^3$, if $\psi = e^{i\alpha}\phi$ for some phase factor $e^{i\alpha}$. The problem now is, what is the quotient space? (Whatever it is, it will be the space of physical spin states for an electron, and it will also be the complex projective space $\mathbb{C}P^1$.) The space of phase factors is a circle (S^1), so we can write this quotient space as

$$\frac{S^3}{S^1}. \tag{5}$$

The set of phase factors also constitutes the group $U(1)$, so people would often write this with $U(1)$ in the denominator instead of S^1. In either case, to make the notation precise it has to be understood precisely what the equivalence relation is.

Now for the problem. This whole problem is DTB. If you know already that $\mathbb{C}P^1$ is S^2 and how this is interpreted as the space of physical spin states of a spin-$\frac{1}{2}$ particle, you need not do this problem.

(a) Let ψ be a normalized, 2-component spinor, an element of $S^3 \subset \mathbb{R}^4 = \mathbb{C}^2$, and consider the equivalence class of normalized spinors $[\psi]$ containing ψ, where two normalized spinors are considered equivalent if they differ by an overall phase. Show that the equivalence class $[\psi] \subset S^3$ is a circle. This means, show that there is a bijection from this equivalence class to a circle. **Hint:** Show that the map,

$$f : S^1 \to S^3 : e^{i\alpha} \mapsto e^{i\alpha}\psi \tag{6}$$

is an injection, where S^1 means the unit circle in the complex plane and where ψ is a given, normalized spinor (therefore, a point on S^3). Also note that for any injective map, if the range is restricted to the image of the map, then the map is automatically a bijection.

Since equivalence classes always divide a space into disjoint subsets, this shows that S^3 can be divided into a family of circles. (And the decomposition is continuous.) This cannot be done with

S^2.

(b) Consider the map,

$$\pi : S^3 \to S^2 : \psi \mapsto \langle\psi|\boldsymbol{\sigma}|\psi\rangle, \tag{7}$$

where $\boldsymbol{\sigma}$ are the Pauli matrices and again $\psi \in S^3$ because it is a normalized spinor. Show that $\langle\psi|\boldsymbol{\sigma}|\psi\rangle$ is a unit vector in $\mathbb{R}^3$, therefore an element of S^2. Call this unit vector $\hat{\mathbf{n}}$; it is the direction the spinor is "pointing in." We want to show that that this map π is the projection map for the phase equivalence relation, which will show that

$$\frac{S^3}{S^1} = S^2. \tag{8}$$

(c) To do this we must show that π is a bijection between the set of equivalence classes and S^2. Part (b) has already shown that π maps into S^2 (that is, S^2 can be taken as the range of π). Show that if $\psi \sim \phi$, that is, $\psi = e^{i\alpha}\phi$ for some phase $e^{i\alpha}$, then $\pi(\psi) = \pi(\phi)$. Thus, the inverse image of a point of S^2 under π consists of one or more equivalence classes, so that π can be regarded as a map from equivalence classes into S^2.

(d) Show that if ψ and ϕ are not equivalent, then $\pi(\psi) \neq \pi(\phi)$. This means that π, regarded as a map between equivalence classes and S^2, is injective.

(e) Show that π, considered as a map from equivalence classes to S^2, is surjective. This means that for every $\hat{\mathbf{n}} \in S^2$, there exists a $\psi \in S^3$ such that $\pi(\psi) = \hat{\mathbf{n}}$. **Hint:** Start with the spinor "pointing in" the positive z-direction. Apply rotation operators to make it "point in" the direction with spherical coordinates (θ, ϕ). Show that π applied to this spinor gives $\hat{\mathbf{n}}$ also pointing in the direction given by (θ, ϕ).

Since π is both injective and surjective, it is a bijection, and there is a one-to-one correspondence between phase equivalence classes and points of S^2. This proves (8). The foliation of S^3 into circles, whose quotient space is S^2, is called the *Hopf fibration.* It is very important in the representation theory of $SU(2)$, in the theory of spin coherent states, spin networks and many other areas. It was shown in class that the sphere S^2 is the set of polarization states of electromagnetic waves. The Hopf fibration also appears in the classical mechanics of the two-dimensional harmonic oscillator, in which the energy shell in the 4-dimensional phase space is S^3, and the classical orbits are circles that foliate S^3 in a Hopf fibration. Also, as far as the mathematics is concerned, we see that $\mathbb{C}P^1 = S^2$.