Physics 250
Fall 2008
Homework 8
Due Friday, October 24, 2008

Reading Assignment: Nakahara, pp. 207-227. See also Frankel, pp. 80–109, 391–410.

Notes. On p. 204, Nakahara defines a "volume form" as an m-form on an m-dimensional manifold that vanishes nowhere, and he discusses the theorem that such a form exists iff the manifold is orientable. I'd prefer it if he did not call this a "volume form," since I'm afraid that word suggests the usual volume computed with a metric. There is no metric in Nakahara's construction. If the manifold does have a metric, then the volume is defined in a unique way (whereas Nakahara's "volume forms" are far from unique). Nakahara uses his "volume forms" in his development of the subject of integration. I have done it a different way in lecture.

On p. 206, Nakahara introduces the "partition of unity," a useful concept theoretically, but I have skipped it in lecture because we can get by without it. See the discussion in Frankel.

On p. 208, Nakahara has an exercise to show that $O(1,3)$ (the Lorentz group) has four connected components. This was mentioned in class (the four components are resolved by parity and time reversal). The identity component consists of the "proper" Lorentz transformations, the ones that do not reverse time or change the orientation of the spatial axes. This group (the identity component) has a double cover representing Lorentz transformations on spinors, which turns out to be $SL(2,\mathbb{C})$. The 4-component Dirac spinors are Lorentz transformed by a direct sum of two inequivalent representations of $SL(2,\mathbb{C})$.

On p. 210, Nakahara's Eq. (5.112) is meaningless, at least in the center term, since there is no meaning to the bracket of vectors at a specific point $g \in G$ (unless $g = e$). This is just sloppy notation in proving the theorem, which is done right in the notes.

On p. 212, Nakahara's definition of a 1-parameter subgroup is fine, it is a homomorphism $\phi : \mathbb{R} \to G$ (ϕ was denoted σ in the lectures). But in Eq. (5.118) he wants to define a vector field associated with the 1-parameter subgroup as the vector field that has the 1-parameter subgroup as an integral curve. The problem with this is that you cannot define a *field* by means of a single *integral curve*, which in general does not explore the whole manifold. You can use a single integral curve to define a vector at each point of the curve, but it doesn't make a whole field. So at this point I stop reading up through Eq. (5.122). This subject is treated more carefully without such sloppy reasoning in the lecture notes.

On p. 213, the paragraph beginning with "Conversely, ..." is covering the same territory covered in lecture, but I think it's done more clearly in lecture. Nakahara's notation $\sigma(t,g)$ means the same thing as $\Phi_t\, g$ in the lecture notes. When he writes,

$$\frac{d\sigma(t,g)}{dt} = X, \tag{7.1}$$

what he means is that X is the vector field whose advance map is σ (his notation) or Φ (mine). This is the general relation between vector fields and advance maps (on any manifold). I would write it this way,

$$\frac{d}{dt}\Big|_{t=0} \Phi_t^* = X, \tag{7.2}$$

where $X \in \mathfrak{X}(M)$, $\Phi_t : M \to M$ is the advance map, and both sides are understood to act on $\mathfrak{F}(M)$. Again, I think this material is covered more clearly in the notes.

On p. 216, Nakahara defines the action of a Lie group on a manifold. His map $\sigma : G \times M \to M$ is what I called Φ in lecture. Actually, I've tended to use the alternative notation, $\Phi(g, x) = \Phi_g\, x$, where $x \in M$, so that $\Phi_g : M \to M$. The remark at the top of p. 217 is a warning about an upcoming abuse of notation, in which $\Phi_g\, x$ is simply written gx.

On p. 224, Nakahara calls ad_a the "adjoint representation". I don't know anyone else who uses that terminology. I used the notation I_a for what he calls ad_a (it is the "inner automorphism" action). What he calls the "adjoint map" is what most people call the "adjoint representation."

On pp. 226–227, Nakahara distinguishes between the standard simplex $\bar{\sigma}$ and any old simplex σ, but I can't see how he ever uses this distinction in his approach to integration theory. Also, given a map $f : \sigma \to M$ of a simplex into a manifold, he refers to the image of σ (he must mean the image of f) as the "singular simplex." Most authors would use the term "singular simplex" to refer to the pair, (σ, f). Actually the image of f, as a subset of M, is not so important for the theory of integration, mainly because everything gets pulled back to $\mathbb{R}^m$ using f^*.

The normal approach to the integration of forms is to define use a standard region in $\mathbb{R}^m$, a simplex, cube, polyhedron, etc., which is mapped into the manifold by some f. The result is called a "singular" simplex, cube, polyhedron, etc, where "singular" means that f is not necessarily injective nor is f_* of maximal rank. (The electron can stop for lunch for an hour if it wants.)

1. In class we developed the differential geometric theory of Lie algebras of a group by using left-invariant vector fields. But one can also define right-invariant vector fields. Do these give a different definition of the Lie algebra of a group (that is, of the Lie algebraic structure on $T_eG = \mathfrak{g}$)? In this problem we explore this question.

(a) A right-invariant vector field is defined by $X_V^R|_a = (R_{a*}|_e)V$, where the R superscript means "right," and $a \in G$, $V \in \mathfrak{g}$. When necessary to distinguish right- and left-invariant vector fields, we will write X_V^R and X_V^L, otherwise we will assume X_V (without a superscript) is left-invariant. Let $\sigma^R(t)$ be the integral curve of X_V^R passing through e at $t = 0$. Show that $\sigma^R(t) = \sigma^L(t)$, where $\sigma^L(t) = \sigma(t)$ is the integral curve of the left-invariant vector field X_V^L, passing through e at $t = 0$, which was discussed in lecture.

Hint: Do this by showing that $X_V^R = X_V^L$ when evaluated at a point on the integral curve $\sigma^L(t) = \sigma(t)$. The easiest way I found to do this was to represent a vector at a point by an equivalence class of curves $[c]$, and to note that the tangent map F_* can be defined by $F_*[c] = [F \circ c]$.

Thus, we can drop any superscript and just write $\sigma(t) = \exp(tV)$, as in the lecture notes. Find an expression for other integral curves of X_V^R (with other initial conditions) in terms of $\exp(tV)$.

(b) Suppose we define a new bracket $[\,,\,]^R$ of elements $V, W \in \mathfrak{g}$ by writing

$$[V, W]^R = [X_V^R, X_W^R]|_e. \tag{7.3}$$

What is the relation between this bracket and the bracket defined in class (which used left-invariant vector fields)?

Hint: Think about the geometrical meaning of the Lie bracket in terms of the commutativity of flows. By this time you should know how to compute flows for arbitrary initial conditions for both left- and right-invariant vector fields, in terms of $\exp(tV)$.

2. The configuration space of a rigid body with one point fixed is $SO(3)$. To establish a one-to-one correspondence between configurations in a physical sense and matrices $A \in SO(3)$ we proceed as follows. Let the fixed point of the rigid body be the origin of a set of right-handed, inertial coordinates. Let $\{\hat{\mathbf{e}}_i, i = 1, 2, 3\}$ be the unit vectors of this coordinate system. We call $\{\hat{\mathbf{e}}_i\}$ the *space frame*. Let $R : \mathbb{R}^3 \to \mathbb{R}^3$ be a proper rotation leaving the origin fixed. We associate R with a matrix $A \in SO(3)$ by writing,

$$A_{ij} = \hat{\mathbf{e}}_i \cdot (R\hat{\mathbf{e}}_j). \tag{7.4}$$

Choose a reference orientation for the rigid body, and let the mass density be $\rho(\mathbf{x})$ in the reference orientation. The reference orientation can be chosen arbitrarily; it need not be an initial condition. The actual orientation at some time t is specified by a matrix $A(t) \in SO(3)$, which rotates the reference orientation to the actual orientation. Thus, the orbit of the classical system is a trajectory $A(t)$ through $SO(3)$ that satisfies the equations of motion.

Define time-dependent unit vectors by

$$\hat{\mathbf{f}}_i(t) = R(t)\hat{\mathbf{e}}_i. \tag{7.5}$$

The vectors $\{\hat{\mathbf{f}}_i\}$ constitute the *body frame*. You can think of $\{\hat{\mathbf{e}}_i\}$ as being imbedded in the body when the body is in its reference orientation, and $\{\hat{\mathbf{f}}_i\}$ as being fixed in the body in its actual orientation, rotating with the body.

The moment of inertia tensor in the body frame is the same as the moment of inertia tensor in the space frame when the body is in the reference orientation. It is given by

$$M_{ij} = \int d^3\mathbf{x}\, \rho(\mathbf{x})(|\mathbf{x}|^2\delta_{ij} - x_i x_j). \tag{7.6}$$

The moment-of-inertia tensor has components which are time-independent in the body frame, but the space frame components are time-dependent.

The angular velocity $\boldsymbol{\omega}$ is defined by

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{x}, \tag{7.7}$$

where $\mathbf{x}$ is the position of a particle fixed in the body, and $\mathbf{v}$ is its velocity. This equation can be referred either to the body frame or the space frame.

The kinetic energy of the rigid body is

$$T = \frac{1}{2}\boldsymbol{\omega} \cdot M \cdot \boldsymbol{\omega}, \tag{7.8}$$

where $\boldsymbol{\omega}$ is the angular velocity and where it is understood that both $\boldsymbol{\omega}$ and M are taken with respect to the same frame (space or body).

(a) Nakahara's matrices X_x, X_y, X_z on the bottom of p. 223 span the Lie algebra of $SO(3)$. For this problem, call these matrices $\{J_i, i = 1, 2, 3\}$, but remember that the subscript on J_i is a label of the matrix, not a component index. The actual components of these matrices are

$$(J_i)_{jk} = -\epsilon_{ijk}. \tag{7.9}$$

Recall that in class, a tangent vector X to the group manifold of a real matrix group was associated with a matrix X (same symbol) by writing,

$$X = \sum_{ij} X_{ij} \frac{\partial}{\partial x_{ij}}, \tag{7.10}$$

where $\{x_{ij}\}$ are the components of the matrix (hence coordinates on matrix space). For example, if evaluated at the identity, this matrix represents an element of the Lie algebra. For practice you should work out the matrix representing a left-invariant (or right-invariant) vector field, evaluated at an arbitrary point (matrix) of the group.

For the rigid body problem, the velocity of the system is specified by a matrix $\dot{A}$. Express this as a linear combination of the left-invariant vector fields, using the basis $\{J_i\}$ of the Lie algebra, and interpret the components in ordinary (non-differential geometric) language. Do the same for the right-invariant vector fields.

(b) For two dimensional rotations, we can write

$$\omega = \frac{d\theta}{dt}, \tag{7.11}$$

where $\theta(t)$ is the angle of rotation. In three dimensions, we can write the angular velocity in terms of the Euler angles (θ, ϕ, ψ) and their derivatives. The actual formulas for the space components of the angular velocity are

$$\begin{aligned} \omega_x &= -\sin\phi\,\dot{\phi} + \cos\phi\sin\theta\,\dot{\psi}, \\ \omega_y &= \cos\phi\,\dot{\theta} + \sin\phi\sin\theta\,\dot{\psi}, \\ \omega_z &= \dot{\phi} + \cos\theta\,\dot{\psi}, \end{aligned} \tag{7.12}$$

but you do not need to do anything with these equations except to note that they are complicated and not very symmetrical. Nevertheless, compared to the two-dimensional case, the question arises

whether these equations are messy because of the definitions of the Euler angles. Maybe with some other definition of angles, call them $\boldsymbol{\theta} = (\theta_x, \theta_y, \theta_z)$, we would have an obvious generalization of the two-dimensional formula,

$$\boldsymbol{\omega} = \frac{d\boldsymbol{\theta}}{dt}. \tag{7.13}$$

Can such angles $\boldsymbol{\theta}$ be found? Explain why or why not.

(c) In last week's homework, you worked out the Euler-Lagrange equations for a mechanical system when the velocity is expressed in a non-coordinate basis. You should have obtained,

$$\frac{d\pi_\mu}{dt} = e^\nu_\mu \frac{\partial \bar{L}}{\partial x^\nu} - c^\sigma_{\mu\nu}\, v^\nu \pi_\sigma, \tag{7.14}$$

(See last week's homework for the notation).

For the force-free rigid body, the Lagrangian is just the kinetic energy. Use Eq. (7.14) to obtain the equations of motion for the force-free rigid body. Do this in the body frame.

3. Induced vector fields were discussed in class. We are given an action of a Lie group G on a manifold M by means of diffeomorphisms $\Phi_g : M \to M$. We let $V \in \mathfrak{g}$. We associate V with a vector field $V_M \in \mathfrak{X}(M)$ by writing,

$$V_M = \left.\frac{d}{dt}\right|_{t=0} \Phi^*_{\exp(tV)}, \tag{7.15}$$

where it is understood that both sides act on $\mathfrak{F}(M)$. More explicitly, this is

$$(V_M f)(x) = \left.\frac{d}{dt}\right|_{t=0} f(\Phi_{\exp(tV)} x), \tag{7.16}$$

for all $x \in M$, $f \in \mathfrak{F}(M)$.

Since we are using the symbol Φ for the action of G on M, let us use the symbol Ψ for advance maps. If X is a vector field (on any manifold), let the corresponding advance map be denoted by $\Psi_{X,t}$. The relation between the advance map and the vector field is

$$X = \left.\frac{d}{dt}\right|_{t=0} \Psi^*_{X,t}, \tag{7.17}$$

where again both sides are acting on scalar fields.

Returning to the induced vector field V_M, note that its integral curves are given by the action of the 1-parameter subgroup $\exp(tV)$ on an initial point, that is,

$$\Psi_{V_M,t}\, x = \Phi_{\exp(tV)}\, x. \tag{7.18}$$

(a) If $V, W \in \mathfrak{g}$, express the Lie bracket $[V_M, W_M]$ in terms of the Lie algebra bracket $[V, W]$. Hint: I found it useful to introduce the notation,

$$K_x : G \to M : g \to \Phi_g x, \tag{7.19}$$

and then to use K_x^* to pull back functions from M to G, where brackets can be evaluated. Note that $\operatorname{im} K_x$ is the orbit of the group action through x.

(b) Find the induced vector fields for the actions $g \mapsto L_g$ and $g \mapsto R_{g^{-1}}$ of G on itself.

(c) The adjoint representation (my terminology, not Nakahara's) is the linear representation of G acting on its own Lie algebra, $g \mapsto \mathrm{Ad}_g$, where $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$ is a linear map defined by

$$\mathrm{Ad}_g W = (I_{g*}|_e)W. \tag{7.20}$$

Here I_g is the inner automorphism, $I_g : G \to G : a \mapsto gag^{-1}$. The infinitesimal generator of this action is a vector field on $\mathfrak{g}$. However, since $\mathfrak{g}$ is a vector space, the value of the vector field at any point $W \in \mathfrak{g}$ can be regarded as just another vector in $\mathfrak{g}$, parameterized by the point W at which the field is evaluated. That is, the vector can be translated parallel to itself to move its base to the origin. With this understanding, show that

$$\mathrm{ad}_V W = \left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}_{\exp(tV)} W = [V, W], \tag{7.21}$$

where $\mathrm{ad}_V W$ is standard notation for this infinitesimal generator. As you see, it is an alternative notation for the bracket of elements of $\mathfrak{g}$.