Physics 250
Fall 2008
Homework 7
Due Friday, October 17, 2008

Reading Assignment: Nakahara, pp. 196–206. See also Frankel, pp. 52–84, 125–136, 145–154.

Notes. Equation (5.83), p. 202, is meaningless. Please ignore it. Also, Eq. (5.89), p. 203, should read $\theta = p_\mu \, dq^\mu$. Otherwise the material on pp. 191–204 is ok.

1. (DTB) Nakahara Exercise 5.15, p. 199 (Exercise 5.32, p. 161 of the first edition).

2. (DTB) As was discussed in class, the Lie derivative L_X obeys the Leibnitz rule when acting on tensor products. As was also discussed, the exterior product $\wedge$ is an antisymmetrized tensor product.

(a) Let $\alpha \in \Omega^r(M)$ and $\beta \in \Omega^s(M)$. Find and expression for $L_X(\alpha\wedge\beta)$ in terms of $L_X\alpha$ and $L_X\beta$.

(b) The *Cartan formula* is

$$L_X = i_X d + d i_X, \tag{7.1}$$

where $X \in \mathfrak{X}(M)$, valid when both sides act on differential forms. Show that the right hand side obeys the same rule when acting on $\alpha\wedge\beta$ as does L_X in part (a).

(c) Show that the Cartan formula (7.1) is valid when acting on 0-forms and 1-forms.

(d) Explain why parts (a)–(c) prove the Cartan formula in all cases (that is, when acting on arbitrary differential forms).

3. As explained in the notes, a *symplectic manifold* M is a manifold endowed with a 2-form ω such that $d\omega = 0$ (ω is closed) and $\det \omega_{\mu\nu} \neq 0$ (ω is nondegenerate; $\omega_{\mu\nu}$ is the component matrix of ω in some chart). Such an ω is referred to as a *symplectic 2-form.* The nondegeneracy condition can be stated in a coordinate-free manner by saying that $\omega(X,Y) = 0$ for all Y iff $X = 0$. Here ω means ω evaluated at some point $z \in M$, $X, Y \in T_zM$, and the condition is to hold at all $z \in M$.

Since $\det \omega_{\mu\nu} \neq 0$, the Poisson tensor with components $J^{\mu\nu}$, defined by

$$J^{\mu\nu}\omega_{\nu\alpha} = \delta^\mu_\alpha, \tag{7.2}$$

is defined. It can be used to compute Poisson brackets by

$$\{A, B\} = A_{,\mu} \, J^{\mu\nu} \, B_{,\nu}. \tag{7.3}$$

In these equations, components are taken with respect to an arbitrary coordinate system (call it z^μ) on the symplectic manifold. Do not assume that these coordinates are the usual (q_i, p_i) coordiantes of a mechanical system, indeed, the definition of a symplectic manifold says nothing about the existence of such coordinates.

Prove the Jacobi identity for the Poisson bracket,

$$\{\{A, B\}, C\} + \{\{B, C\}, A\} + \{\{C, A\}, B\} = 0. \tag{7.4}$$

It will probably be easiest to do this in coordinates.

Although the definition of a symplectic manifold says nothing about the existence of canonical (q, p) coordinates, it can be shown that such coordinates always exist locally on a symplectic manifold. This is Darboux's theorem. But do not use this theorem in proving Eq. (7.4), that is, Darboux's theorem is too big a hammer to smash a small nut, namely, the Jacobi identity. Just do the calculation in an arbitrary coordinate system.

4. Hamiltonian mechanics in noncanonical coordinates. Let $P = \mathbb{R}^6$ be the phase space of a charged particle of charge e moving in a magnetic field $\mathbf{B}(\mathbf{x})$, with coordinates $q_i = x_i$ and p_i, $i = 1, 2, 3$, where p_i is the canonical momentum of the particle, given in terms of its velocity by

$$\mathbf{p} = m\Big(\mathbf{v} + \frac{e}{c}\mathbf{A}(\mathbf{x})\Big). \tag{7.5}$$

Use coordinates $z^\mu = (\mathbf{x}, \mathbf{v})$ on phase space. Write the symplectic form ω in terms of the differentials dx_i, dv_i. Write the Hamiltonian as a function of $(\mathbf{x}, \mathbf{v})$. Translate Hamilton's equations,

$$i_X\omega + dH = 0, \tag{7.6}$$

into these coordinates, and solve for $\dot{\mathbf{x}}$, $\dot{\mathbf{v}}$.

This calculation sheds light on why the usual Hamiltonian for a particle in a magnetic field looks so complicated. It is because the proper way to view the effect of a magnetic field on the classical dynamics is to say that it modifies the symplectic form, not the Hamiltonian.