Physics 250
Fall 2008
Homework 6
Due Friday, October 10, 2008

Reading Assignment: Nakahara, pp. 184–195. See also Frankel, pp. 58–64, 125–136.

1. Nakahara, Exercises 5.3 and 5.5, p. 187 (Exercises 5.13, 5.15, pp. 148–149 of the first edition).

2. In this problem we consider the behavior of vector fields and advance maps under diffeomorphisms. The background is that we are given a vector field X on a manifold M, and a diffeomorphism $f : M \to N$. As explained in class, since f is a diffeomorphism, $f_* X$ is a vector field on N ($X \in \mathfrak{X}(M)$ and $f_* X \in \mathfrak{X}(N)$). We wish to show that advance maps commute with the action of f, that is,

$$f \circ \Phi_t = \Psi_t \circ f, \tag{6.1}$$

where $\Phi_t : M \to M$ is the advance map for X, and $\Psi_t : N \to N$ is the advance map for $f_* X$.

Let $\sigma : [a, b] \to M$ be an integral curve of X passing through $x_0 \in M$ at $t = 0$, and let $\tau : [a, b] \to N$ be given by $\tau = f \circ \sigma$. Show that τ is an integral curve of $f_* X$ passing through $y_0 = f(x_0)$ at $t = 0$. Do this in some local coordinates (say x^i on M, and y^i on N). This proves Eq. (6.1). This is an example of a proof that is easy in coordinates, but more taxing when put into coordinate-free language.

3. (DTB) The *Kronecker tensor* δ is a type $(1, 1)$ tensor. Make sure you understand the difference between a tensor at a point $x \in M$ and a tensor field. δ at a point $x \in M$ is a map,

$$\delta|_x : T^*_x M \times T_x M \to \mathbb{R}, \tag{6.2}$$

whereas as a field δ is a map,

$$\delta : \mathfrak{X}^*(M) \times \mathfrak{X}(M) \to \mathfrak{F}(M) : (\alpha, Y) \mapsto \alpha(Y), \tag{6.3}$$

where the final formula defines δ. Let $X \in \mathfrak{X}(M)$, and compute $L_X \delta$ (the Lie derivative of δ along X).

4. A problem on coordinate and noncoordinate bases. Only part (a) is marked DTB.

(a) (DTB) Let x^μ be the coordinates in a chart on manifold M. Inside the domain of the chart, the vector fields $\{\partial/\partial x^\mu\}$ form a basis in the tangent spaces to M, that is, if these vector fields are evaluated at a point $p \in M$, then they form a set of $m = \dim M$ linearly independent vectors in $T_p M$, for each p in the domain of the chart. These vector fields are said to form a *coordinate basis.*

Let $\{e_\mu, \mu = 1, \ldots, m\}$ be a new set of vector fields that are also linearly independent at each point p in some region contained in the domain of the chart x^μ. Notice that the μ subscript on e_μ is not a component index, rather it serves to label the vector fields. The e_μ can be expanded as linear combinations of the coordinate basis vectors,

$$e_\mu = e_\mu^\nu \frac{\partial}{\partial x^\nu}, \tag{6.4}$$

where e_μ^ν are the expansion coefficients. These are functions of position in M.

Show that the set $\{e_\mu\}$ is also a coordinate basis, that is, that there exist scalars y^μ such that

$$e_\mu = \frac{\partial}{\partial y^\mu}, \tag{6.5}$$

if and only if

$$[e_\mu, e_\nu] = 0, \tag{6.6}$$

where $[\,,\,]$ is the Lie bracket. This is a local construction.

Whether or not the set $\{e_\mu\}$ is a coordinate basis, the Lie brackets of the basis fields among themselves are interesting. These Lie brackets are themselves vector fields, and so can be expanded as linear combinations of the basis vectors. That is, an expansion of the form,

$$[e_\mu, e_\nu] = c^\sigma_{\mu\nu}\, e_\sigma \tag{6.7}$$

exists. The expansion coefficients $c^\sigma_{\mu\nu}$ are called the *structure constants*, although in this context they are not constant (they are functions of position).

(b) Let M be the configuration space of a mechanical system in classical mechanics, with coordinates x^μ imposed (in some chart). These are what are called "generalized coordinates" in classical mechanics, meaning that they are not necessarily Cartesian coordinates (nor for that matter is M necessarily a vector space $\mathbb{R}^m$).

A configuration point at position $x \in M$ may be given any velocity. The combination of the position and velocity of the configuration constitutes what we call the *dynamical state* of the system, because the knowledge of the dynamical state allows us to determine the subsequent evolution. The velocity is specified by the time derivatives $\dot{x}^\mu$, which can be regarded as the components in the coordinate basis of the velocity vector $V \in T_x M$,

$$V = \dot{x}^\mu \frac{\partial}{\partial x^\mu}. \tag{6.8}$$

The *state space* of the system is the space of all possible dynamical states, that is, all possible configurations with all possible velocities at a given configuration. This space is the tangent bundle TM. (See the lecture notes for the definition of TM.)

The Lagrangian is usually regarded as a function $L(x^\mu, \dot{x}^\mu)$, but we can see it abstractly as a scalar $L : TM \to \mathbb{R}$. The equations of motion (the Euler-Lagrange equations) are

$$\frac{dp_\mu}{dt} = \frac{\partial L}{\partial x^\mu}, \tag{6.9}$$

where p_μ is the *canonical momentum*, defined by

$$p_\mu = \frac{\partial L}{\partial \dot{x}^\mu}. \tag{6.10}$$

Suppose we write the tangent vector V as a linear combination of some other basis $\{e_\mu\}$ (besides the coordinate basis $\{\partial/\partial x^\mu\}$),

$$V = \dot{x}^\mu \frac{\partial}{\partial x^\mu} = v^\mu \, e_\mu, \tag{6.11}$$

which defines the components v^μ with respect to the new basis (itself not necessarily a coordinate basis). Then we can transform the Lagrangian from the variables $\dot{x}^\mu$ to the variables v^μ. We write this tranformation,

$$L(x^\mu, \dot{x}^\mu) = \bar{L}(x^\mu, v^\mu), \tag{6.12}$$

using a new symbol $\bar{L}$ to indicate that the Lagrangian has been expressed in terms of new variables. Also, let us define

$$\pi_\mu = \frac{\partial \bar{L}}{\partial v^\mu}, \tag{6.13}$$

which as it turns out are the components of the momentum with respect to the basis dual to $\{e_\mu\}$.

Find a nice equation of evolution for $d\pi_\mu/dt$ in terms of the derivatives of the Lagrangian $\bar{L}$ and the structure constants of the basis.