Physics 250
Fall 2008
Homework and Notes 3
Due Friday, September 17, 2008

Reading Assignment: Nakahara, Chapter 3. I also recommend Frankel, pp. 333–354.

1. Read the notes below.

(a) Let G be a free, finitely generated Abelian group of rank 2, which according to the theorems is isomorphic to $\mathbb{Z}^2$. Identify the generators (x_1, x_2) with the vectors $(0,1)$ and $(1,0)$. Let H be the subgroup of G that is generated by $(2,6) \in \mathbb{Z}^2$. Find generators (y_1, y_2) of G such that H is generated by a multiple of y_1.

(b) Show that every row and every column of an element of $GL(r,\mathbb{Z})$ is relatively prime (the integers have no common factors except ± 1).

Notes. The following are some notes on the subject of finitely generated Abelian groups, which one must work with to compute homology groups over the integers $\mathbb{Z}$. In the following we quote some theorems without proof.

Finitely generated Abelian groups are important in our approach to homology theory because the triangulations that we use to study the topology of a manifold have only a finite number of faces, edges, vertices, etc. As was explained this week, faces, edges, vertices, etc. are technically examples of *oriented simplexes* of different dimensionalities. For example, the "edges" of our triangulation will be oriented 1-simplexes. With a motivation that comes from line, surface, volume, etc. integrals, we also consider linear combinations of these 1-simplexes with integer coefficients, as objects of various dimensionalities that we might integrate over. Such objects are called *chains*. The fact that we are using integer coefficients in forming our chains means that the homology groups we ultimately derive will be considered to be over the integers $\mathbb{Z}$. (But later we will want to think about homologies over other sets of coefficients, such as $\mathbb{R}$ or $\mathbb{Z}_2$, the integers modulo 2.)

Thus, the set of chains (of a given dimensionality) that we will consider consists of all linear combinations with integer coefficients of some set of simplexes (of the given dimensionality). A given chain can be identified by a vector of integers, the coefficients $(n_1, \ldots, n_r)$, where r is the number of "basis" simplexes. The space of such chains is $\mathbb{Z}^r$, which can be viewed geometrically as an integer lattice in r-dimensional space. This set is not a vector space according to the technical definition of a vector space (because $\mathbb{Z}$ is not a field), but it obviously has many of the properties of a vector space (you can add chains, the rule is just the addition of integer vectors). Technically, the set of chains (of a given dimensionality) is best regarded as an *Abelian group*, in which the zero chain, corresponding to vector $(0, \ldots, 0)$, is the identity and the "multiplication law" is vector addition.

The set of chains of dimensionality k (in some triangulation of a manifold M) is denoted $C_k(M)$. Based on what has been said, it is clear that $C_k(M)$ is isomorphic to $\mathbb{Z}^r$, where r is the number of k-simplexes in the triangulation. We also want to consider the set of k-dimensional chains that are cycles, denoted by $Z_k(M)$, and the set of k-dimensional chains that are boundaries, denoted by $B_k(M)$. As explained in class, $Z_k(M)$ is a subgroup of $C_k(M)$, and $B_k(M)$ is a subgroup of $Z_k(M)$, that is,

$$B_k(M) \subseteq Z_k(M) \subseteq C_k(M), \tag{1}$$

with each subset relation actually meaning "subgroup." Each of these subgroups can be thought of geometrically as sublattices of the group it is contained in. The homology groups we will be interested in are quotient groups,

$$H_k(M) = Z_k(M)/B_k(M), \tag{2}$$

and, as explained in class, they are independent of the triangulation, that is, they are topological invariants.

So we turn to the theory of Abelian groups. First, we define a *finitely generated* Abelian group as one for which every $g \in G$ can be written in the form,

$$g = \sum_{i=1}^{s} n_i x_i, \tag{3}$$

where $(x_1, \ldots, x_s)$ is a list of elements of G and $n_i \in \mathbb{Z}$. Notice that there is no attempt to say that s is "minimal" in any sense, or that the x_i are "linearly independent." In fact, some of the x_i could be zero, or duplicates, or "linear combinations" of others, insofar as this definition is concerned. Given a finitely generated Abelian group, the generators are not unique, in fact, even their number is not unique. In the following discussion (and in our treatment of homology) we will only be interested in finitely generated Abelian groups.

There are two kinds of finitely generated Abelian groups, those that are *free* and those that are not free. If every element $g \in G$ (henceforth assumed to be Abelian and finitely generated) for some choice of generators $(x_1, \ldots, x_r)$ can be written *uniquely* in the form

$$g = \sum_{i=1}^{r} n_i x_i, \tag{4}$$

for some integer vector $(n_1, \ldots, n_r) \in \mathbb{Z}^r$, then the group is said to be *free* and of *rank* r. For the case of a free group, the generators that enter into the definition of freeness are fixed in number (that is, the number r, the rank, is a fixed characteristic of the group). The free group itself is isomorphic to $\mathbb{Z}^r$. The chain group $C_k(M)$ is an example of a free Abelian group.

For a free, finitely generated Abelian group, the generators $(x_1, \ldots, x_r)$ are not unique, even if their number (r) is. Given one set of generators, we can create another set that will work just as well, by writing

$$y_i = \sum_{j=1}^{r} M_{ij}\, x_j, \tag{5}$$

where the $r \times r$ matrix M must belong to the group $GL(r, \mathbb{Z})$. This group is defined as the group of all $r \times r$ integer matrices that have an integer matrix as an inverse. An integer matrix belongs to this group if and only if its determinant is ± 1.

Theorem. *Every subgroup of a free, finitely generated Abelian group is a free, finitely generated Abelian group.*

Let G be a free, finitely generated Abelian group of rank r, which therefore is isomorphic to $\mathbb{Z}^r$, and let H be a subgroup. Geometrically, H is a sublattice of G. Think, for example, of the case $r = 1$, so that $G \cong \mathbb{Z} = \{\ldots, -1, 0, 1, 2, \ldots\}$, and let $H = \{\ldots, -2, 0, 2, 4, \ldots\}$. That is, let H consist of the even integers. This is a sublattice of the one-dimensional lattice. Nakahara denotes this H by $2\mathbb{Z}$. As an abstract group, H is also isomorphic to $\mathbb{Z}$, but as a subgroup of G, it contains only one half of G. For the case $r = 2$, for practice you may try drawing a 2-dimensional integer lattice (for G) and then pick out a pair of "basis vectors" that "span" (generate) a 2-dimensional sublattice (for H), but one that is not the whole lattice G. For the case $r = 2$ other possible subgroups H are one-dimensional sublattices. And the "zero-dimensional" sublattice is just the trivial subgroup containing the identity, $H = \{0\}$. If you think about these examples, then the theorem above should be plausible. You will also see that if the rank of G is r, then the rank of its subgroup H is some integer p with $0 \leq p \leq r$.

This theorem implies that both $Z_k(M)$ and $B_k(M)$ are free, finitely generated Abelian groups (because they are subgroups of $C_k(M)$). Geometrically, they can be seen as sublattices of $\mathbb{Z}^r$, where $C_k(M) \cong \mathbb{Z}^r$.

The quotient group of a free, finitely generated Abelian group G and one of its subgroups (which must be free) is not necessarily free. You can see this already in the one-dimensional case, in which

$$\frac{\mathbb{Z}}{k\mathbb{Z}} = \mathbb{Z}_k = \{0, \ldots, k-1\}, \tag{6}$$

where the quotient group is the "cyclic" group of order k (so called because if you take the generator and keep adding it to itself, you come back to 0 periodically). The quotient group is not a lattice, but rather a discrete analog of a circle. The following theorem generalizes this one-dimensional case to arbitrary dimensions.

Theorem. *Let G be a free, finitely generated Abelian group of rank r, and let H be a subgroup. Then it is always possible to choose the generators $(x_1, \ldots, x_r)$ of G so that every element $h \in H$ can be written uniquely in the form,*

$$h = \sum_{i=1}^{p} n_i k_i x_i, \tag{7}$$

where $p \leq r$, $n_i \in \mathbb{Z}$, and $k_i \geq 1$. Thus, $H \cong \mathbb{Z}^p$, and H is generated by $(k_1 x_1, \ldots, k_p x_p)$ (integer multiples of the first p generators of G), and H is of rank p. The case $p = 0$ means that H is the trivial subgroup $\{0\}$.

This theorem contains the previous one, but provides more information. The next theorem follows rather easily from it.

Theorem. *Let G be a free, finitely generated Abelian group of rank r, and let H be a subgroup. Then*

$$\frac{G}{H} \cong \mathbb{Z}_{k_1} \times \ldots \times \mathbb{Z}_{k_p} \times \mathbb{Z} \times \ldots \times \mathbb{Z}, \tag{8}$$

where $k_i \geq 1$ and there are $r - p$ final factors of $\mathbb{Z}$.

In fact the form given by this theorem for G/H is the general form for an arbitrary (free or non-free) finitely generated Abelian group. To see this, let A be an arbitrary (free or non-free) finitely generated Abelian group, with generators $(x_1, \ldots, x_s)$, and remember how nonunique the generators are. Nevertheless, since they are generators, every element $a \in A$ can be written in the form

$$a = \sum_{i=1}^{s} n_i x_i. \tag{9}$$

This equation can be regarded (for given generators) as specifying a map $f : \mathbb{Z}^s \to A$. This map is clearly onto, so $\operatorname{img} f = A$. The kernel of this map, the set of integer vectors $(n_1, \ldots, n_s)$ in $\mathbb{Z}^s$ that map onto $0 \in A$, constitute a subgroup of $\mathbb{Z}^s$. Therefore the quotient group $\mathbb{Z}^s/(\ker f)$, is isomorphic to A (by the theorem proved in class on kernels and images of group homomorphisms). But with $\mathbb{Z}^s$ identified with G and $\ker f$ identified with H in the theorem above, this shows that A is isomorphic to some number of products of $\mathbb{Z}_k$ with $k \geq 1$ times some number of product of $\mathbb{Z}$. If a factor $\mathbb{Z}_k$ with $k = 1$ occurs, then this factor can be discarded (as far as abstract groups are concerned), because $\mathbb{Z}_1$ is just the trivial group $\{0\}$. If you have redundant generators x_i in the original set of generators, or if you have one set of generators and then throw in some more that are dependent on the ones already given, then this just produces more factors $\mathbb{Z}_1$ in the final product. The number of factors of $\mathbb{Z}$ in the final product, however, is independent of the choice of generators.

Thus, an arbitrary (free or non-free) finitely generated Abelian group can be seen geometrically as a kind of a discrete analog of a cylinder (some number of lines crossed with some number of circles), sort of a multidimensional nanotube. Homology groups, being quotient groups of a free Abelian group with a subgroup, are always of this form.

2. Nakahara, problem 3.1, p. 120 (p. 87 in the first edition).

3. Nakahara, problem 3.2, p. 120 (p. 88 in the first edition).

More notes. Nakahara's Chapter 3 has many defects in its details, which you will see if you read the chapter carefully. If you find yourself confused by something he says, don't assume that it is your fault. Ask me about it if you want to be sure.

Note that Nakahara often writes 0 when he means $\{0\}$, that is, the Abelian group containing just one element. This leads to confusing looking things like $0/0$, when he means $\{0\}/\{0\}$. The latter is perfectly well defined. Each group, the numerator and the denominator, has one element, the identity, so the quotient group has just one element, too, the coset which is the whole group. The quotient group is also the trivial Abelian group, so $\{0\}/\{0\} \cong \{0\}$.

Nakahara writes $\oplus$ when I would write $\times$, for example, his $\mathbb{Z} \oplus \mathbb{Z}$ is what I would write as $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$ (it is the set of ordered pairs of integers, (n_1, n_2)).

Nakahara's Lemma 3.2 and Theorem 3.2 (p. 97) are confusing because he never mentions that you have to change basis to bring the generators in the form he discusses, nor does he discuss the nonuniqueness in the generators. This is taken care of by the notes above. Some of the other theorems in the chapter are awkwardly done, for example, Theorem 3.5, p. 110.