Physics 250

Fall 2008

Homework and Notes 2

Due Friday, September 12, 2008

Reading Assignment:

1. (DTB) On p. 77 Nakahara talks about the orthogonal complement to the kernel of f. The objection to this, as discussed in class, is that you can't define an orthogonal complement without a metric, which you may not have (and which Nakahara has not yet introduced at that point in the text). Another problem with Nakahara's remark is that the space spanned by the vectors h_i (his notation) is not unique, because the h_i are not (in general) unique.

Let V be a vector space and U a vector subspace. Let $v_1 \sim v_2$ if $v_1 - v_2 \in U$. Geometrically, this means that v_1 and v_2 lie in a "plane" parallel to U. Show that $\sim$ is an equivalence relation. Let $V/\sim$ be denoted V/U. Show that V/U can be given the structure of a vector space (one can define addition of equivalence classes and their multiplication by scalars).

If we now let $f : V \to W$ be a linear map and identify U with $\ker f$, there is an obvious way to define a mapping

$$\hat{f} : \frac{V}{\ker f} \to \operatorname{im} f. \tag{1}$$

Do this, and show that the mapping is a vector space isomorphism. In an appropriate basis, $\hat{f}$ is represented by a square, invertible matrix, even though f originally may well have been represented by a noninvertible matrix, even a rectangular matrix. It is in this sense that all matrices have an inverse, even rectangular ones. But note that the domain of $\hat{f}$ is not a *subspace* of V, it is a *quotient space*.

If a metric is introduced into V, so that orthogonality is defined, then show how the subspace of V which is orthogonal to $\ker f$ can be identified with the quotient space $V/\ker f$.

2. Let V be a vector space and $U \subseteq V$ be a vector subspace. Let V^* be the dual space to V. Let $X^* \subseteq V^*$ be the space of dual vectors that annihilate U, that is, $\alpha \in X^*$ if $\alpha(u) = 0$ for all $u \in U$. Prove that

$$\dim U + \dim X^* = \dim V. \tag{2}$$

If now we have a mapping $f : V \to W$, show that

$$\dim \operatorname{im} f = \dim \operatorname{im} f^*, \tag{3}$$

where $f^* : W^* \to V^*$ is the pull-back.

A remark here is that if we have a subspace $U \subseteq V$, one way to specify U is to specify a set of vectors that span U (a basis in U). But a complementary way is to specify a complete set of covectors that annihilate U (a basis in X^*, in the notation above). U is then the simultaneous kernel of these covectors. This is an example of switching from a space to its dual to understand a problem, often an effective strategy.

3. (DTB) This problem concerns the adjoint $\tilde{f}$ of a linear operator $f : V \to W$ between two vector spaces V and W (both over field K). It is assumed that V possesses metric g, and W possesses metric G. In class it was explained that the adjoint $\tilde{f} : W \to V$ can be defined as

$$\tilde{f} = g^{-1} f^* G, \tag{4}$$

where f^* is the pull-back and the metrics g and G are seen as maps between the vector spaces V and W and their duals. Another interpretation of the metrics is in terms of scalar products, and it gives another way to define $\tilde{f}$, namely,

$$\langle \tilde{f} w, v \rangle_g = \langle w, f v \rangle_G, \tag{5}$$

for all $v \in V$ and all $w \in W$. That is, in Eq. (5), f is assumed given and $\tilde{f}$ is defined by that equation as the unique linear operator : $W \to V$ that satisfies the given condition. Show that definitions (4) and (5) are equivalent.

4. In class we used the 2-torus T^2 on which to draw examples of 1-cycles that are or are not homologous. You may have noticed in these examples that homologous 1-cycles can be continously deformed into one another, and 1-cycles that are boundaries can be continuously contracted to a point. These are special features of a torus that make it a bad example, because homology does not have anything to do with continous deformations or contractions. The latter belong rather to *homotopy* theory.

Find (that is draw) a 2-dimensional compact manifold with a cycle on it that is a boundary but is not contractible to a point. Find one with two homologous 1-cycles that cannot be continuously deformed into one another.