**Physics 250**
**Fall 2008**
**Homework 15**
**Due Friday, December 12, 2008**

**Reading Assignment:** Nakahara, pp. 374–385.

**1.** (DTB) Nakahara problem 9.2, p. 372 (problem 2, p. 328, of the first edition).

**2.** (DTB) A problem on the Frobenius theorem. As discussed in class, a $k$-*distribution* over a manifold $M$ ($\dim M = m$) is an assignment of $k$-dimensional subspaces ($k \leq m$) in the tangent spaces $T_x M$ for all $x \in M$ (or perhaps only over some region $U \in M$). The assignment is assumed to be smooth. We say that a $k$-distribution (in some region) is *integrable* if there exists an $(m-k)$-parameter family of $k$-dimensional surfaces in the region which are everywhere tangent to the distribution. The surfaces are only required to exist locally.

Let $X$, $Y$ be vector fields on $M$ which lie in a $k$-distribution $\Delta$. If $\Delta$ is integrable, then the integral curves of both $X$ and $Y$ must lie in the surfaces tangent to $\Delta$. Therefore if we follow the integral curves of $X$ or $Y$ in any order for any elapsed parameters, we must always remain on a given surface. Considering infinitesimal elapsed parameters, we see that the Lie bracket $[X, Y]$ must be tangent to the surface, that is, it must lie in the distribution. Thus, we have a theorem, that if $\Delta$ is integrable and $X, Y \in \Delta$, then $[X, Y] \in \Delta$. It turns out the converse is also true (at least locally): if for every vector fields $X, Y \in \Delta$, we have $[X, Y] \in \Delta$, then $\Delta$ is (locally) integrable.

Let $\theta^\alpha$, $\alpha = 1, \ldots, m-k$ be a set of linearly independent 1-forms that annihilate $\Delta$. Show that the following four conditions are equivalent:

**(i)** For every $X, Y \in \Delta$, $[X, Y] \in \Delta$.

**(ii)** $d\theta^\alpha(X, Y) = 0$ for all vector fields $X, Y \in \Delta$.

**(iii)** There exist 1-forms $\lambda^\alpha{}_\beta$ such that $d\theta^\alpha = \lambda^\alpha{}_\beta \wedge \theta^\beta$.

**(iv)** $d\theta^\alpha \wedge \Omega = 0$, where $\Omega = \theta^1 \wedge \ldots \wedge \theta^{m-k}$.