Physics 250
Fall 2008
Unhomework 14
Nothing due

Notes. In Sec. 9.2.3 on bundle maps, a little thought will show that just preserving fibers is not a strong enough condition on a map between fiber bundles that is supposed to define a kind of a bundle homomorphism. (It's obvious that a proper definition must also involve the group and the transition functions.) The proper definition is given in Steenrod. I never used this concept in lecture.

In lecture I presented an alternative definition of the pullback of a bundle, from that used by Nakahara in Sec. 9.2.5. His definition is clever geometrically, but I think not very motivated. I will make it a special case of the reconstruction of a bundle.

Nakahara's statement at the bottom of p. 356, about π_2 having maximal rank m, makes no sense. π_2 maps a $2m$-dimensional manifold to another one, so the maximal rank is $2m$. I do not know what he is trying to say in this section. If anyone can figure it out, please let me know.

On p. 357, Nakahara never actually says what the "homotopy axiom" is. But the theorem that a bundle is trivial if the base space is contractible is an important one, and worthy of a special name.

I shall skip the material on the canonical line bundle, since we skipped Chapter 8.

On p. 359, in the first paragraph of Sec. 9.3.2 (on Frames), Nakahara seems to be saying that the components of the basis vectors are "unit vectors" in their own basis. This is a trivial statement of linear algebra and rather pointless in this context.

I will probably skip the material on Whitney sum bundles and tensor product bundles in lecture, due to lack of time.

On p. 363, the fact that the structure group has a natural action on every principal fiber bundle is important. I gave a several motivating examples in lecture because the general statement of this fact (Nakahara's Eq. (9.41), essentially reproduced in lecture) is (to me) rather unmotivated.

On p. 364, second paragraph, he means $s_i(p)$ instead of $s_1(p)$. If you have a principal fiber bundle, then, as pointed out in class, the fibers are diffeomorphic to the structure group, but they are not groups since they have no preferred origin. But if you choose an origin by some prescription, then you automatically get a specific identification of the fiber with the structure group. This is what a local section does: it picks out one point on each fiber (over the local U_i), which serves as an "origin" for the fiber. Conversely, if you use the local trivialization ϕ_i, with constant group element (say, $g = e$), then this maps U_i onto P creating a local section. This is what Nakahara calls the "canonical local section." I'd prefer not to call it "canonical", since anything that depends on the local trivialization is highly arbitrary.

Nakahara's definition of the Hopf map would look simpler to the eye of a physicist if he changed the sign on Eq. (9.53b) and wrote the result in the form, $\boldsymbol{\xi} = \langle z|\boldsymbol{\sigma}|z\rangle$, as discussed in class. As we say, $\boldsymbol{\xi}$ is the "direction" the spinor z is "pointing in." This is common language in quantum mechanics, but it only works for a spin 1/2 particle (for other spins, the expectation value of the spin operator is not in general a unit vector, and it does not specify the spinor, even to within a phase).

Notice that an example of Eq. (9.64) (for $n = 3$) was proved in a homework. The proof for general n is similar.

On p. 370, Nakahara gives a definition of a fiber bundle associated with a principal fiber bundle as $(P \times F)/G$ (the action of G on $P \times F$ is given by his Eq. (9.66)). In class I explained associated bundles in terms of the reconstruction program. I think I know why Nakahara does it his way, it leads to a useful point of view in Kaluza-Klein theories, for example. But for now I did not want to multiply constructions when one would do.

On p. 371, Nakahara mentions the obstruction to the construction of the spin bundle. This concerns the possibility of defining Dirac or Weyl spinors over a topologically nontrivial space-time (for example, with worm holes). It involves the cohomology classes of the space-time manifold, not over $\mathbb{R}$ but over $\mathbb{Z}_2$. This topic is given a nice discussion by Frankel.