Physics 221A
Fall 1996
Final Exam
Tuesday, December 17, 1996

This exam has 100 points total.

1. (25 = 7+3+7+8) At the crudest level of description, the Hamiltonian for the hydrogen atom is

$$H_0 = \frac{p^2}{2m} - \frac{e^2}{r}.$$

At this level of description, the energy levels, degeneracies, and allowed electric dipole transitions are indicated schematically in the diagram below. As you can see, the degeneracies include the electron spin degrees of freedom. The selection rules used in the diagram below are $\Delta\ell = \pm 1$.

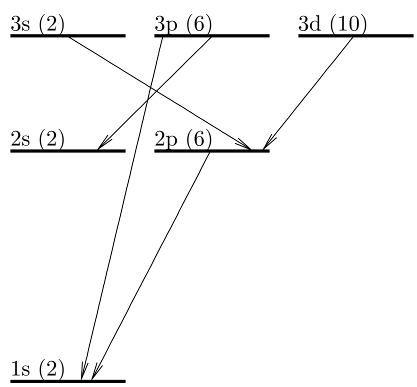

Figure for Problem 1.

(a) (7 points) Redraw this diagram to indicate the changes which occur when the three fine structure terms (spin-orbit, Darwin, and relativistic kinetic energy correction) are added to H_0. Label the new levels by standard notation (e.g., $1s_{1/2}$), indicate degeneracies and allowed electric dipole transitions. Of course your diagram will be schematic and not to scale, but if you want two or more states to have the same energy, you should make it clear that they are on the same horizontal level. Indicate what selection rules you are using to find the allowed transitions.

(b) (3 points) Redraw the $n = 2$ levels when the effects of the Lamb shift are included.

(c) (7 points) Redraw the $n = 1$ and $n = 2$ levels when hyperfine effects are included. Think of ordinary hydrogen with the proton for a nucleus. Supply all relevant quantum numbers and degeneracies (now including the proton spin degrees of freedom). Indicate the allowed dipole transitions from the $2p$ states to the $1s$ states, and indicate which selection rules you are using. If you can't remember what order the energy levels occur within a hyperfine multiplet, just guess.

(d) (8 points) Magnetic dipole transitions in hydrogen are governed by matrix elements of the operator $\mathbf{L} + 2\mathbf{S}$. (The transition rate is proportional to the square of such matrix elements, taken between the initial and final atomic states.) In the Dirac picture, in which the eigenstates of hydrogen have the form $|n\ell j m_j\rangle$, what are the selection rules for magnetic dipole transitions? The transition $2s \to 1s$ is forbidden as an electric dipole transition. Is it allowed as a magnetic dipole transition? Higher order multipole transitions (electric quadrupole, etc.) are governed by matrix elements of irreducible tensor operators T^k_q, with $k \geq 2$. For which values of k is the $2s \to 1s$ transition allowed?

2. (20 = 6+7+7) In helium electric dipole transitions are governed by the matrix element $\langle b|\mathbf{D}|a\rangle$, where

$$\mathbf{D} = -e(\mathbf{r}_1 + \mathbf{r}_2).$$

(a) (6 points) Prove that transitions between ortho and para states (intercombination lines) are forbidden.

(For this to be strictly true we must ignore spin-orbit effects. I don't expect you to become confused over this point since we did not consider spin-orbit effects in helium.)

(b) (7 points) Give *brief* physical explanations for the following facts concerning the structure of helium: (1) Within a configuration $1s\,n\ell$, the energy is an increasing function of ℓ; (2) within the configuration $1s\,n\ell$, the para states are higher in energy than the ortho states; (3) the $1s^2$ configuration has no ortho state; (4) there are no states with configuration $2s^2$.

(c) (7 points) How would the spectrum of helium be different if the electron were a spin-0 boson? Indicate whether the bound state energy levels would shift (it is not necessary to be quantitative, but indicate whether the levels go up or down); indicate whether some levels disappear or new levels appear; and indicate what happens to the degeneracies of the levels. Justify (*briefly!*) your answers.

3. (10 points) When we combine two identical $s = \frac{1}{2}$ particles, we get a total spin of either $S = 0$ (singlet) or $S = 1$ (triplet). The triplet states are even under spin exchange, while the singlet states are odd. Similarly, when we combine two identical $s = 1$ particles, we get $S = 0, 1, 2$, where $S = 2$ is even under exchange, $S = 1$ is odd, and $S = 0$ is even. These examples suggest (correctly) that when we combine two identical particles of arbitrary s, the stretched states $S = 2s$ are even under spin exchange, the next set of states $S = 2s - 1$ are odd, etc.

Consider two identical particles with spin s, and let $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$. The usual eigenstates of S^2 and S_z are $|SM\rangle$, which are given by

$$|SM\rangle = \sum_{m_1 m_2} |sm_1\rangle |sm_2\rangle \langle ssm_1 m_2 | SM\rangle.$$

In a product of kets like $|sm_1\rangle|sm_2\rangle$, we assume that the first ket belongs to particle 1, and the second to particle 2. Define the spin exchange operator by

$$E_{12}|sm_1\rangle|sm_2\rangle = |sm_2\rangle|sm_1\rangle.$$

Prove that $|SM\rangle$ is an eigenstate of E_{12} (for any values of S and M), and that the eigenvalue does not depend on M. Hint: If you try to prove this by using identities involving the Clebsch-Gordan coefficients, then you are on the wrong track.

4. (10 points) It is claimed that an arbitrary rotation R can be written in the form,

$$\mathsf{R} = \mathsf{R}(\hat{\mathbf{z}}, \alpha)\mathsf{R}(\hat{\mathbf{y}}, \beta)\mathsf{R}(\hat{\mathbf{z}}, \gamma),$$

for some α, β, γ. Find α, β, γ for $\mathsf{R}(\hat{\mathbf{x}}, \pi/2)$.

5. (10 points) For the electronic structure of the hyrogen molecule ion H_2^+, we consider the motion of one electron in the field of two protons, which we imagine to be pinned down at locations $z = \pm a$ on the z-axis, as indicated in the figure below.

The Hamiltonian is

$$H = \frac{p^2}{2m} - \frac{e^2}{\sqrt{x^2 + y^2 + (z-a)^2}} - \frac{e^2}{\sqrt{x^2 + y^2 + (z+a)^2}},$$

which includes all the gross Coulomb effects but ignores fine structure, etc. Since the Hamiltonian does not depend on spin, we treat the electron as if it were spinless.

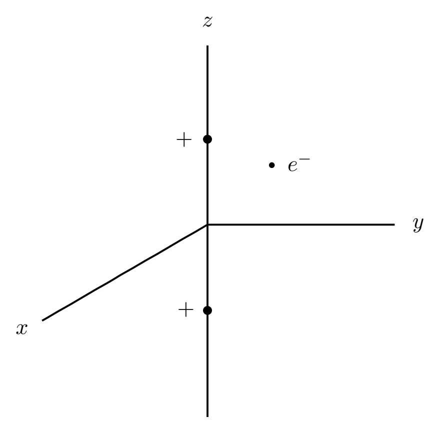

Figure for Problem 5.

List all the obvious symmetries of this Hamiltonian, i.e., list all the operators (either symmetry operators or generators of symmetry operators) which commute with H.

In most problems in the real world, there are no degeneracies beyond those which can be explained by the obvious symmetries. This is true for H_2^+. There is a certain obvious class of eigenstates of this system which are degenerate, and a certain class which are nondegenerate. Explain what these classes are, what the orders of the degeneracies are, and explain why there are degeneracies.

6. (25 points) Consider the linear Stark effect on the $n = 3$ levels of hydrogen. Denote the electric field by $\mathbf{F}$ (to avoid confusion with the energy E), and suppose it is directed in the z-direction, $\mathbf{F} = F\hat{\mathbf{z}}$. You can ignore the electron spin, and treat the $n = 3$ level as if it were 9-fold degenerate before the electric field is turned on. Without evaluating any radial or angular integrals, show that the energy shifts have the form,

$$\Delta E = 0, \quad \pm\sqrt{A^2 + B^2}, \quad \pm\frac{\sqrt{3}}{2}B,$$

where A and B are quantities which you must define but which you need not evaluate. Indicate the degeneracies of these levels. Hint: The problem involves a 3×3 matrix. Do this one first. You may use the table of Clebsch-Gordan coefficients.