Physics 221A
Fall 2020
Appendix D
Vector Calculus†

1. Introduction

In the following we let f be a scalar field and $\mathbf{A}$ a vector field.

2. Differential Operators in Rectangular Coordinates

The components of $\mathbf{A}$ are given by

$$\mathbf{A} = A_x\,\hat{\mathbf{x}} + A_y\,\hat{\mathbf{y}} + A_z\,\hat{\mathbf{z}}. \tag{1}$$

Then

$$\nabla f = \frac{\partial f}{\partial x}\,\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\,\hat{\mathbf{y}} + \frac{\partial f}{\partial z}\,\hat{\mathbf{z}}, \tag{2}$$

$$\nabla\times\mathbf{A} = \Big(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\Big)\,\hat{\mathbf{x}} + \Big(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\Big)\,\hat{\mathbf{y}} + \Big(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\Big)\,\hat{\mathbf{z}}, \tag{3}$$

$$\nabla\cdot\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}, \tag{4}$$

and

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}. \tag{5}$$

3. Differential Operators in Cylindrical Coordinates

Cylindrical coordinates are (ρ, ϕ, z), defined by

$$\begin{aligned} x &= \rho\cos\phi, \\ y &= \rho\sin\phi, \\ z &= z. \end{aligned} \tag{6}$$

The orthonormal frame of associated unit vectors is

$$\begin{aligned} \hat{\boldsymbol{\rho}} &= \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}, \\ \hat{\boldsymbol{\phi}} &= -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}, \\ \hat{\mathbf{z}} &= \hat{\mathbf{z}}, \end{aligned} \tag{7}$$

† Links to the other sets of notes can be found at:
`http://bohr.physics.berkeley.edu/classes/221/1920/221.html`.

with

$$\hat{\boldsymbol{\rho}}\times\hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}. \tag{8}$$

Define the components of **A** by

$$\mathbf{A} = A_\rho\,\hat{\boldsymbol{\rho}} + A_\phi\,\hat{\boldsymbol{\phi}} + A_z\,\hat{\mathbf{z}}. \tag{9}$$

Then

$$\nabla f = \frac{\partial f}{\partial \rho}\,\hat{\boldsymbol{\rho}} + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\,\hat{\boldsymbol{\phi}} + \frac{\partial f}{\partial z}\,\hat{\mathbf{z}}, \tag{10}$$

$$\nabla\times\mathbf{A} = \Big(\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\Big)\hat{\boldsymbol{\rho}} + \Big(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\Big)\hat{\boldsymbol{\phi}} + \frac{1}{\rho}\Big[\frac{\partial}{\partial \rho}(\rho A_\phi) - \frac{\partial A_\rho}{\partial \phi}\Big]\hat{\mathbf{z}}, \tag{11}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}, \tag{12}$$

and

$$\nabla^2 f = \frac{1}{\rho}\frac{\partial}{\partial \rho}\Big(\rho\frac{\partial f}{\partial \rho}\Big) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}. \tag{13}$$

Also, the following Jacobians are sometimes useful:

$$\frac{\partial(xyz)}{\partial(\rho\phi z)} = \begin{pmatrix} \cos\phi & -\rho\sin\phi & 0 \\ \sin\phi & \rho\cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{14}$$

where the rows are labeled by (xyz) and columns by $(\rho\phi z)$, for example, $\partial x/\partial\phi = -\rho\sin\phi$. The inverse Jacobian is

$$\frac{\partial(\rho\phi z)}{\partial(xyz)} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\dfrac{\sin\phi}{\rho} & \dfrac{\cos\phi}{\rho} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{15}$$

4. Differential Operators in Spherical Coordinates

Spherical coordinates are (r,θ,ϕ), defined by

$$\begin{aligned} x &= r\sin\theta\cos\phi, \\ y &= r\sin\theta\sin\phi, \\ z &= r\cos\theta. \end{aligned} \tag{16}$$

The orthonormal frame of associated unit vectors is

$$\begin{aligned} \hat{\mathbf{r}} &= \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}, \\ \hat{\boldsymbol{\theta}} &= \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}}, \\ \hat{\boldsymbol{\phi}} &= -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}, \end{aligned} \tag{17}$$

with

$$\hat{\mathbf{r}}\times\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}. \tag{18}$$

Define the components of $\mathbf{A}$ by

$$\mathbf{A} = A_r\,\hat{\mathbf{r}} + A_\theta\,\hat{\boldsymbol{\theta}} + A_\phi\,\hat{\boldsymbol{\phi}}. \tag{19}$$

Then

$$\nabla f = \frac{\partial f}{\partial r}\,\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\,\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\,\hat{\boldsymbol{\phi}}, \tag{20}$$

$$\nabla\times\mathbf{A} = \frac{1}{r\sin\theta}\Big[\frac{\partial}{\partial\theta}(A_\phi\sin\theta) - \frac{\partial A_\theta}{\partial\phi}\Big]\,\hat{\mathbf{r}} + \frac{1}{r}\Big[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(rA_\phi)\Big]\,\hat{\boldsymbol{\theta}} + \frac{1}{r}\Big[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\Big]\,\hat{\boldsymbol{\phi}}, \tag{21}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}, \tag{22}$$

and

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\Big(r^2\frac{\partial f}{\partial r}\Big) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\Big(\sin\theta\frac{\partial f}{\partial\theta}\Big) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}. \tag{23}$$

The following Jacobian matrices are sometimes useful:

$$\frac{\partial(xyz)}{\partial(r\theta\phi)} = \begin{pmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix}, \tag{24}$$

and its inverse,

$$\frac{\partial(r\theta\phi)}{\partial(xyz)} = \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \dfrac{\cos\theta\cos\phi}{r} & \dfrac{\cos\theta\sin\phi}{r} & -\dfrac{\sin\theta}{r} \\ -\dfrac{\sin\phi}{r\sin\theta} & \dfrac{\cos\phi}{r\sin\theta} & 0 \end{pmatrix}. \tag{25}$$

5. The Line Element

Let two nearby points have coordinates (x, y, z) and $(x + dx, y + dy, z + dz)$, or (r, θ, ϕ) and $(r + dr, \theta + d\theta, \phi + d\phi)$, etc, depending on the coordinates. Let ds be the distance between the two points. Then, in the various coordinate systems, we have

$$ds^2 = dx^2 + dy^2 + dz^2, \tag{26a}$$

$$= d\rho^2 + \rho^2\,d\phi^2 + dz^2, \tag{26b}$$

$$= dr^2 + r^2\,d\theta^2 + r^2\sin^2\theta\,d\phi^2. \tag{26c}$$

Dividing this by dt^2, we get the square of the velocity, $v^2 = \mathbf{v}\cdot\mathbf{v}$, expressed in terms of the time derivatives of the coordinates. This gives

$$v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2, \tag{27a}$$

$$= \dot{\rho}^2 + \rho^2\,\dot{\phi}^2 + \dot{z}^2, \tag{27b}$$

$$= \dot{r}^2 + r^2\,\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2. \tag{27c}$$

From the components of the line element one can read off the components of the metric tensor.

6. Vector Calculus in Two Dimensions

In two dimensions the usual coordinates are either rectangular, (x, y) or polar (ρ, ϕ). The coordinate transformation and formulas for the unit vectors $(\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\phi}})$ are obtained from Eq. (6) and (7) simply by omitting the terms referring to z or $\hat{\mathbf{z}}$. The formulas for ∇f, $\nabla \cdot \mathbf{A}$ and $\nabla^2 f$ can be obtained from those in Secs. 2 and 3 simply by omitting the terms that refer to z or A_z. As for the curl of a vector field, $\nabla \times \mathbf{A}$, in two dimensions it can be regarded as a scalar that is identified with the z-component of the corresponding three-dimensional formulas, (3) or (11). The line element in two dimensions is given by Eq. (26b), omitting the term referring to z.